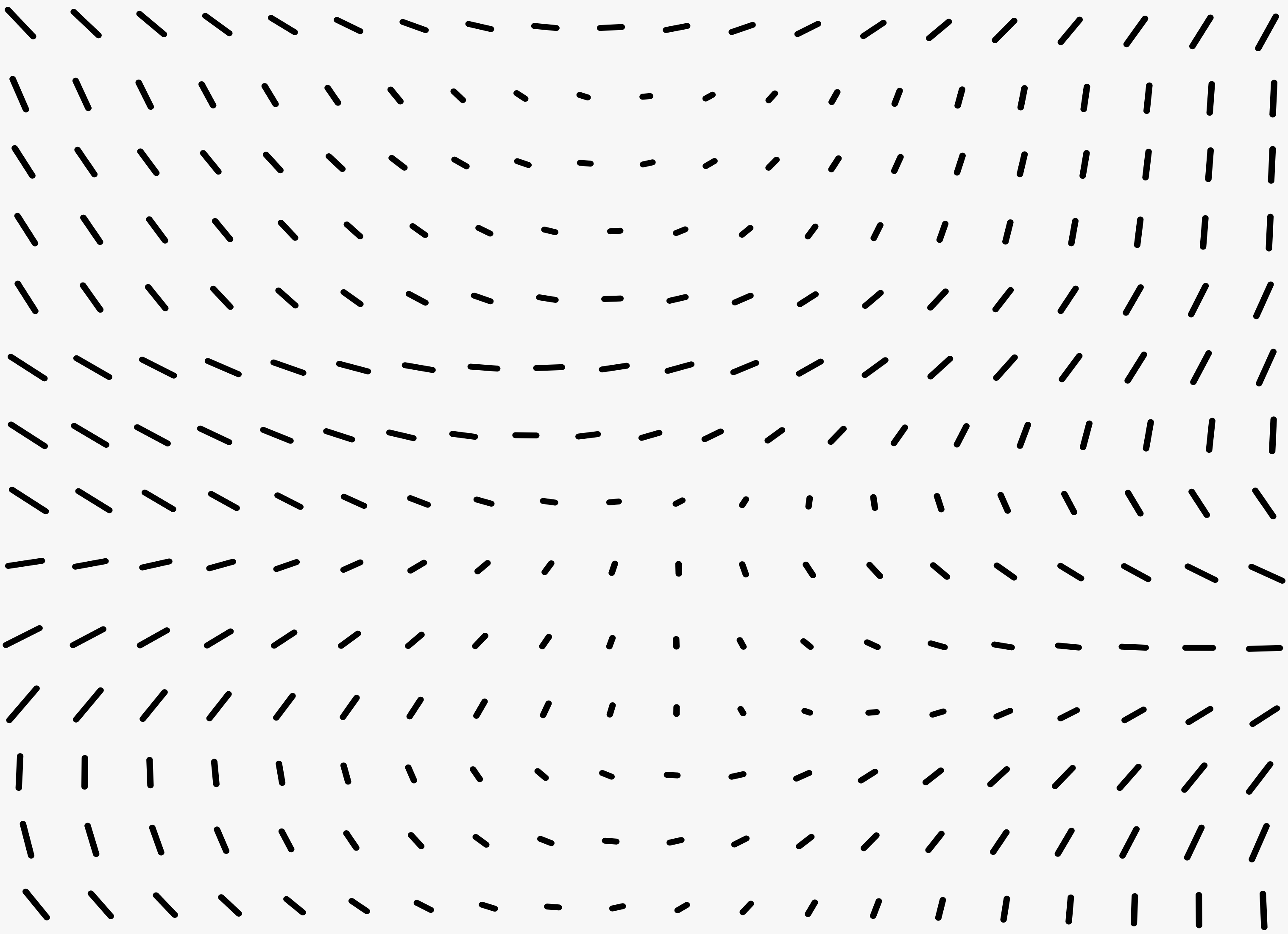
ANALYSIS 1



TRIANGLE INEQ. $\triangleright$ for $x, y \in \mathbb{R}$:

1. $|x+y| \leq |x| + |y|$
2. $|x-y| \geq |x| - |y|$
3. $|x-y| \geq |y| - |x|$

$\left.\begin{array}{l} 2. \\ 3. \end{array}\right\}$ 4. $|x-y| \geq \big||x| - |y|\big|$

BERNOULLI INEQ. $(1+x)^n \geq 1 + nx$, $x \geq -1$, $n > 0$

Some useful results:

· As 3 : $a, b \in \mathbb{R}$, $a < b$, $I = (a,b) \vee (a,b] \vee [a,b) \vee [a,b]$
Then $\sup I = b$, $\inf I = a$

Let $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$ be 2 nonempty sets bounded from above.
$A + B := \{a+b : a \in A, b \in B\}$. $A+B$ is bounded from above and
$\sup(A+B) = \sup(A) + \sup(B)$.

Let $A \subseteq \mathbb{R}$. $-A := \{-a : a \in A\}$.
- u upper bound for $A \iff -u$ lower bound for $-A$
- A bounded from above $\iff -A$ bounded from below
- If A bounded from above, $\inf(-A) = -\sup(A)$

$\mathbb{R} \setminus \mathbb{Q}$ (i.e. irrational numbers) is dense in $\mathbb{R}$ i.e. any interval $(a,b) \in \mathbb{R}$, $a < b$ contains at least 1 irrational number.

· As 4 : A, B are finite nonempty sets.
- $|A| \leq |B| \iff \exists$ an injective function $f : A \to B$
- $|A| \geq |B| \iff \exists$ a surjective function $f : A \to B$

If A is a countably infinite set and $B \subseteq A$, then B is countable.

$\mathbb{N} \times \mathbb{N}$ is countably infinite.

Let A, B be countably infinite sets, $A \cup B$ is countably infinite.

$\mathbb{R} / \mathbb{Q}$ is uncountable.

Let $A_1, A_2, \ldots$ be countably infinite. Then $\bigcup_{n=1}^{\infty} A_i$ is countably infinite.

$|A| = n$. $|\mathcal{P}(A)| = 2^n$

The set of all finite subsets of $\mathbb{N}$ is countably infinite.

COMPLETENESS

Let $S \neq \emptyset$, $S \subseteq \mathbb{R}$. S is bounded if :
- S is bounded from below : $\exists\ u \in \mathbb{R}$ s.t. $\forall x \in S$, $x \geq u$
- S is bounded from above : $\exists\ u' \in \mathbb{R}$ s.t. $\forall x \in S$, $x \leq u'$

Let $S \subseteq \mathbb{R}$. If $m \in \mathbb{R}$ is an upper bound of S s.t. $m \leq m'$ for every upper bound m' of S, then m is the supremum of S ie $\sup S = m$
- supremum $\equiv$ least upper bound

Let $S \subseteq \mathbb{R}$. If $t \in \mathbb{R}$ is a lower bound of S s.t. $t \geq t'$ for every lower bound of S, then t is an infimum of S ie $\inf S = t$.
- infimum $\equiv$ greatest lower bound.

If S has a maximum s (i.e. $s \in S \wedge s \geq x \ \forall x \in S$) then $\sup S = s$.
If S has a minimum s (i.e. $s \in S \wedge s \leq x \ \forall x \in S$) then $\inf S = s$.

Axiom of Completeness: Let $S \neq \emptyset$, $S \subseteq \mathbb{R}$. If S is bounded from above then $\sup S$ exists. Similarly, if S is bounded from below, $\inf S$ exists.

Archimedean Property of $\mathbb{R}$: Let $x \in \mathbb{R}$ be arbitrary. $\exists\ n \in \mathbb{N}$ s.t. $n > x$.
↳ Corollary: Let $x > 0$. $\exists\ n \in \mathbb{N}$ s.t. $\frac{1}{n} < x$

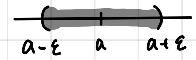
Density of $\mathbb{Q}$ in $\mathbb{R}$: Any interval (a, b), $a, b \in \mathbb{R}$, $a < b$, contains ≥ 1 rational number. Hence we say that $\mathbb{Q}$ is dense in $\mathbb{R}$.
↳ Corollary: any interval (a, b), $a, b \in \mathbb{R}$, $a < b$, contains infinitely many rational numbers.

SEQUENCES

<u>Defn</u> of sequences: A sequence is a function whose domain is $\mathbb{N}$.

Convergence of seq: A seq (a_n) converges to a real number a if, for $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$.

↳ $V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$ is the epsilon-neighbourhood of a.



Outline for convergence proof: $(x_n) \to x$

1. "let $\varepsilon > 0$ be arbitrary"
2. demonstrate a choice for $N \in \mathbb{N}$
3. show that N actually works
4. "Assume $n \geq N$"
5. With N well-chosen, it should be possible to derive the inequality $|x_n - x| < \varepsilon$.

Uniqueness of limits: let (a_n) be a convergent seq, i.e. it has a limit. The limit is unique, i.e. if L_1 and L_2 are limits of (a_n) then $L_1 = L_2$.

- Let (a_n) be a seq, $L \in \mathbb{R}$, (b_n) be a non-negative null seq. If $\exists K \in \mathbb{N}$ s.t. $\forall n \geq K$, $|a_n - L| < b_n$, it follows that (a_n) converges to L.

- All convergent seq of real numbers are bounded.
  ↳ a seq (x_n) is bounded if $\exists M > 0$ s.t. $|x_n| < M$ $\forall n \in \mathbb{N}$.

Algebraic Limit Theorem: $a := \lim(a_n)$, $b := \lim(b_n)$, (a_n) and (b_n) are convergent.

1. $(a_n + b_n) \longrightarrow a + b$
2. $(c \cdot a_n) \longrightarrow ca$, $\forall c \in \mathbb{R}$
3. $(a_n - b_n) \longrightarrow a - b$
4. $(a_n \cdot b_n) \longrightarrow a \cdot b$
5. $\left(\frac{a_n}{b_n}\right) \longrightarrow \frac{a}{b}$ provided that $b \neq 0$.

Limits and order:

Let (a_n) be a convergent seq If $\exists K \in \mathbb{N}$ $\forall n \geq K$: $a_n \geq 0$, then $\lim(a_n) \geq 0$.

- Let (a_n), (b_n) be convergent seq. If $\exists K \in \mathbb{N}$ $\forall n \geq K$: $a_n \leq b_n$ then $\lim(a_n) \leq \lim(b_n)$
  ↳ If $\forall n \geq K$, $a_n < b_n$, we cannot conclude that $\lim(a_n) < \lim(b_n)$. Only that $\leq$.

- Let (b_n) be a seq, $a, c \in \mathbb{R}$. If $\exists K \in \mathbb{N}$ $\forall n \geq K$: $a \leq b_n \leq c$, then $a \leq \lim(b_n) \leq c$.

Squeeze Theorem: Let (a_n), (b_n), (c_n) be seq s.t.

- $\exists K \in \mathbb{N}$ $\forall n \geq K$: $a_n \leq b_n \leq c_n$
- (a_n) and (c_n) converges, $\lim(a_n) = \lim(c_n)$

Then (b_n) converges and $\lim(b_n) = \lim(a_n) = \lim(c_n)$.

Some useful results:

- Lec 9: $\lim(\frac{1}{n}) = 0$
  $\lim(\frac{n}{n^2+1}) = 0$
- Lec 10: $((-1)^n)$ diverges
- Lec 11: for $a > 1$, $(\sqrt[n]{a}) \to 1$
  $\forall k \in \mathbb{N}$, $(\frac{1}{n^k}) \to 0$.
  $\Rightarrow (\frac{a}{n^k}) \to 0$, $a \in \mathbb{R}$
- Lec 12: $\frac{\sin n}{n} \to 0$
- As 5: If $x_n \geq 0 \ \forall n \in \mathbb{N}$ and $\lim(x_n) = 0 \Rightarrow \lim(\sqrt{x_n}) = 0$

  $(\frac{n!}{n^n}) \to 0$

  for $a > 1$, $(\frac{1}{a^n}) \to 0$ also, $(\frac{n}{a^n}) \to 0$
  $0 < a < 1$, $(a^n) \to 0$
  $-1 < a < 0$, $(a^n) \to 0$

  for $0 < a < 1$, $(\sqrt[n]{a}) \to 1$

monotone converges: Let (a_n) be a sequence.
- If $\forall n \in \mathbb{N}$: $a_n \leq a_{n+1}$, (a_n) is monotone increasing.
- If $\forall n \in \mathbb{N}$: $a_n \geq a_{n+1}$, (a_n) is monotone decreasing.

Monotone convergence theorem:
1. Let (a_n) be increasing and bounded from above. Then (a_n) converges and $\lim(a_n) = \sup\{a_n : n \in \mathbb{N}\}$
2. Let (a_n) be decreasing and bounded from below. Then (a_n) converges and $\lim(a_n) = \inf\{a_n : n \in \mathbb{N}\}$

Euler's number: $e := \lim(1 + \frac{1}{n})^n = \lim(1 + \frac{1}{n})^{n+1}$

Some useful results:

- As 6: $(1 + \frac{1}{n})^n > \sqrt{n}$
  $\sqrt[n]{n} < (1 + \frac{1}{n})^2$
  $(\sqrt[n]{n}) \to 1$

  $(1+\frac{1}{n})^{n+1} \to e$
  $(1+\frac{1}{n})^{2n} \to e^2$
  $(1+\frac{1}{n+1})^n \to e$
  $(1-\frac{1}{n})^n \to \frac{1}{e}$

  $(n^{\frac{1}{n^2}}) \to 1$
  $((n!)^{\frac{1}{n^2}}) \to 1$

subsequence: let $n_1, n_2, \ldots \in \mathbb{N}$ s.t. $n_1 < n_2 < \cdots$, and let (a_n) be a sequence.
$(a_{n_k}) = (a_{n_1}, a_{n_2}, \ldots, a_{n_k})$ is a subsequence of (a_n)

- Let (x_n) be a convergent sequence, (x_{n_k}) be an arbitrary subseq of (x_n). Then (x_{n_k}) converges and $\lim(x_{n_k}) = \lim(x_n)$.
  ↳ Corollary:
  Let (x_n) be a sequence and (x_{n_k}), (x_{n_j}) be convergent subsequences, where $\lim(x_{n_k}) \neq \lim(x_{n_j})$. Then, (x_n) diverges.

useful results:

- Lec 13: $(1+\frac{1}{n!})^{n!} \to e$

Bolzano-Weierstrass Theorem: Every bounded seq of $\mathbb{R}$ has a convergent subsequences.

Cauchy sequence: A seq (x_n) is Cauchy iff $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m \geq N : |x_n - x_m| < \varepsilon$

- Every convergent seq is a Cauchy sequence.
- Every Cauchy seq converges.

A seq of $\mathbb{R}$ converges iff it is Cauchy.

- Every Cauchy seq is bounded.

Contractive seq: A seq (x_n) is called contractive if $\exists\ 0 < C < 1$ s.t. $\forall n \in \mathbb{N}$:
$|x_{n+2} - x_{n+1}| \leq C \cdot |x_{n+1} - x_n|$

- Every contractive seq converges.

Steps to find limit or show convergence/divergence for recursively defined (x_n):
1. Prove by induction $\forall x_n$ lie in some bound
2. (x_n) is increasing / decreasing / contractive
3. Apply theorems to show convergence, then apply limit laws to find limit.

Divergence:
- (x_n) diverges to $+\infty$, i.e. $\lim(x_n) = +\infty$, if $\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N : x_n > M$.
- (x_n) diverges to $-\infty$, i.e. $\lim(x_n) = -\infty$, if $\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N : x_n < M$.

- Let (a_n), (b_n) be seq in $\mathbb{R}$, $\lim(b_n) = +\infty$. If $\exists K \in \mathbb{N}$ s.t. $\forall n \geq K : a_n \geq b_n$, then $\lim(a_n) = +\infty$.

Some useful results:
- As 7: $a > 1$, $(\sqrt[n]{a})$ is Cauchy

  (x_n) is a seq in $\mathbb{R}$. If its subseq (x_{2n}), (x_{2n+1}), (x_{3n}) converge, then (x_n) converges.

- As 8: If $(x_n) \in \mathbb{R}$, (x_n) increasing and unbounded, then $\lim(x_n) = +\infty$

- Lec 15:
  $\lim(n) = +\infty$
  $\lim(-n) = -\infty$
  $a > 1$, $\lim(a^n) = +\infty$
  $\lim((1+\frac{1}{n})^{n^2}) = +\infty$

TOPOLOGY

A subset $U \subseteq \mathbb{R}$ is open if $\forall x \in U$, $\exists \varepsilon > 0$ s.t. $V_\varepsilon(x) \subseteq U$.
↳ $\mathbb{R}$ is open.
$\emptyset$ is open.

- Every open <u>interval</u> is open.

- Arbitrary unions of open sets are open. i.e. if I is an arbitrary index set, where $\forall i \in I$: $U_i \subseteq \mathbb{R}$ is open, then $U = \bigcup_{i \in I} U_i$ is also open.

- Finite intersections of open sets are open. i.e. if $U_1, U_2, \dots, U_n \subseteq \mathbb{R}$ are open, then $\bigcap_{i=1}^{n} U_i$ is open.
↳ Infinite intersections of open sets are in general not open.

- A subset of $\mathbb{R}$ is open iff it is a countable union of open sets.

A subset $A \subseteq \mathbb{R}$ is closed if its complement A^c is open.
↳ $\mathbb{R}$ is also closed (since $\mathbb{R}^c = \emptyset$ is open).
Similarly, $\emptyset$ is also closed.

- Every closed interval is closed.

- Finite unions of closed sets are closed. i.e. if $U_1, U_2, \dots, U_n \subseteq \mathbb{R}$ are closed, then $\bigcup_{i=1}^{n} U_i$ is closed.

- Arbitrary intersections of closed sets are closed. Let I be an arbitrary index set, $\forall i \in I$: A_i closed. Then $\bigcap_{i \in I} A_i$ is closed.

Let $A \subseteq \mathbb{R}$. We say that (x_n) is in A if $\forall n \in \mathbb{N}$: $x_n \in A$.

- Let $A \subseteq \mathbb{R}$ be closed and (x_n) is a conv. seq. in A. Then $\lim(x_n) = x \in A$.

Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is a boundary point of A if $\forall \varepsilon > 0$:
$$V_\varepsilon(x) \cap A \neq \emptyset \quad \text{AND} \quad V_\varepsilon(x) \cap A^c \neq \emptyset.$$
↳ set of all boundary points of A is called the boundary of A, ∂A.

- Let $A \subseteq \mathbb{R}$.
  a) A is open iff A does not contain any of its boundary points
  i.e. $A \cap \partial A = \emptyset \Leftrightarrow \partial A \subseteq A^c = \mathbb{R} \setminus A$
  b) A is closed iff A contains all of its boundary points i.e. $\partial A \subseteq A$.

$A \subseteq \mathbb{R}$ is sequentially compact if $\forall$ sequences (x_n) in A, (x_n) has a convergent subseq. (x_{n_k}) s.t. $\lim(x_{n_k}) \in A$.
↳ sequentially compact $\Leftrightarrow$ closed and bounded.

Some results:

- Lec 16: $I = [a, \infty)$, $a \in \mathbb{R}$.

  $\Rightarrow \partial I = \{a\}$

  $\partial[a,b] = \partial[a,b) = \partial(a,b] = \partial(a,b) = \{a,b\}$

- As 8: $\mathbb{Q} \subseteq \mathbb{R}$ is neither open nor closed

  $\partial \mathbb{Q} = \mathbb{R}$

- Tut 9: $\partial A = \partial(A^c)$

LIMITS OF FUNCTIONS

Let $c \in \mathbb{R}$, $\varepsilon > 0$. Then $V_\varepsilon^*(c) := V_\varepsilon(c) \setminus \{c\}$ is the punctured ε-neighborhood of c.

ε-δ definition: Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$. We say that L is the limit of f as $x \to c$ if $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D \setminus \{c\}$ s.t. $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\equiv \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D : \ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\equiv$ " $: \ x \in V_\delta^*(c) \Rightarrow f(x) \in V_\varepsilon(L)$

$\equiv \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D \cap V_\delta^*(c) : f(x) \in V_\varepsilon(L)$

$\equiv \forall \varepsilon > 0 \ \exists \delta > 0 : f(D \cap V_\delta^*(c)) \subseteq V_\varepsilon(L)$

Sequential definition: Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$. We say that L is the limit of f as $x \to c$ if $\forall (x_n)$ in $D \setminus \{c\}$, (x_n) converges with $\lim(x_n) = c \Rightarrow \lim(f(x_n)) = L$.

Some useful results:

Lec 17: $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x$ $\quad c \in \mathbb{R}$ $\quad \lim_{x \to c} f = 2c$

$x \mapsto x^2$ $\quad$ " $\quad \lim_{x \to c} f = c^2$

$f : \mathbb{R} \setminus \{0\}$, $x \mapsto \frac{1}{x}$ $\quad c \in \mathbb{R} \setminus \{0\}$ $\quad \lim_{x \to 0} f = \frac{1}{c}$

Sequential criterion for non existence of the limit of a function:

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$

a) 2 sequence criterion: If $\exists (x_n), (u_n)$ in $D \setminus \{c\}$ s.t. $\lim(x_n) = \lim(u_n) = c$ and both $(f(x_n))$, $(f(u_n))$ converge, BUT $\lim(f(x_n)) \neq \lim(f(u_n))$, then the limit of the function f as $x \to c$ DNE.

b) 1 sequence criterion: If $\exists (x_n)$ in $D \setminus \{c\}$ s.t. $\lim(x_n) = c$ but $(f(x_n))$ diverges, then $\lim f$ as $x \to c$ DNE.

Some useful results:

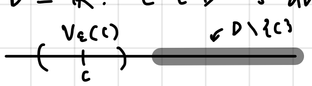
- Lec 18: $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$. $\lim_{x \to 0} f$ DNE.

  Dirichlet function: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ $\quad \lim_{x \to c} f$ DNE.

Let $D \subseteq \mathbb{R}$. A point $c \in \underline{\mathbb{R}}$ is a limit point of D if $\exists (x_n)$ in $D \setminus \{c\}$ s.t. $\lim(x_n) = c$.

A point $c \underline{\in D}$ is an isolated point of D if $\nexists (x_n)$ in $D \setminus \{c\}$ s.t. $\lim(x_n) = c$.

Let $D \subseteq \mathbb{R}$. $c \in D$ is an isolated point of $D \Leftrightarrow \exists \varepsilon > 0 : V_\varepsilon(c) \cap D \setminus \{c\} = \emptyset$.



e.g. 1. $D = \mathbb{N}$. All points in D are isolated.

2. $D = (0, 1]$. Every point in D is a limit point, and so is 0.
   0 is a limit point since $\lim(\frac{1}{n}) = 0$ and $(\frac{1}{n})$ is a seq in D.

3. $D = [1, 2] \cup \{0\}$. 0 is an isolated point, all points in $[1, 2]$ are limit points.

Let $f: D \to \mathbb{R}$, $c \in \mathbb{R}$ s.t. $\lim_{x \to c} f$ exists.

a) If c is a limit point of D, then $\lim_{x \to c} f$ is uniquely determined.

i.e. if $\lim_{x \to c} f = L_1$, $\lim_{x \to c} f = L_2$, $L_1 = L_2$.

b) If c is an isolated point of D, then any $a \in \mathbb{R}$ is a limit of f at c.

The ε-δ definition and sequential defns of the limit of a function are equivalent.

Algebraic Limit Laws: Let $f, g: D \subseteq \mathbb{R} \to \mathbb{R}$, c is a limit point of D. $\lim_{x \to c} f$ and $\lim_{x \to c} g$ exist. Then:

a) $\lim_{x \to c} (f + g) = \lim_{x \to c} f + \lim_{x \to c} g$

b) $\lim_{x \to c} (f - g) = \lim_{x \to c} f - \lim_{x \to c} g$

c) $\lim_{x \to c} (f \cdot g) = \lim_{x \to c} f \cdot \lim_{x \to c} g$

d) $\forall k \in \mathbb{R}: \lim_{x \to c} (k \cdot f) = k \cdot \lim_{x \to c} f$

e) If $\forall x \in D: g(x) \neq 0 \wedge \lim_{x \to c} g \neq 0$, then $\lim_{x \to c} \frac{f}{g} = \lim_{x \to c} f / \lim_{x \to c} g$.

Squeeze Theorem: Let $f, g, h: D \subseteq \mathbb{R} \to \mathbb{R}$. Let c be a limit point of D.
Let $\forall x \in D: f(x) \leq g(x) \leq h(x)$, and $\lim_{x \to c} f = \lim_{x \to c} h = L$.
Then $\lim_{x \to c} g$ exists and $\lim_{x \to c} g = L$.

useful results:

- Lec 19: $\lim_{x \to 0} |x| = 0$

  $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

CONTINUITY

Limit defn: Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $c \in D$. f is continuous at c if $\lim_{x \to c} f = f(c)$.

ε-δ defn: Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $c \in D$. f is continuous at c if
$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D: \; |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.
$\equiv \; \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in V_\delta(c) \cap D: \; f(x) \in V_\varepsilon(f(c))$
$\equiv \; \forall \varepsilon > 0 \; \exists \delta > 0: \; f(V_\delta(c) \cap D) \subseteq V_\varepsilon(f(c))$.

sequential defn: Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $c \in D$. f is continuous at c if
$\forall (x_n)$ in D with $\lim(x_n) = c$, it holds that $\lim(f(x_n)) = f(c)$.

All 3 of these defns are equivalent.

Sequential criterion for discontinuity: Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $c \in D$. If $\exists (x_n)$ in D with $\lim(x_n) = c$ such that:
a) $(f(x_n))$ diverges, OR
b) $(f(x_n))$ converges but $\lim(f(x_n)) \neq f(c)$,
then f is discontinuous at c.

Algebraic Continuity Theorem: Let $f, g: D \subseteq \mathbb{R} \to \mathbb{R}$. $c \in D$. f, g continuous at c. Then:
a) $f + g$ continuous at c
b) $f - g$ continuous at c
c) $\forall k \in \mathbb{R}: k \cdot f$ cont. at c.
d) $f \cdot g$ cont. at c
e) If $\forall x \in D$, $g(x) \neq 0$, then $\frac{f}{g}$ cont. at c.

Let $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$. $f(A) \subseteq B$. Let $c \in A$, $d := f(c)$. Let f be cont at c AND g cont. at d. Then $g \cdot f: A \to \mathbb{R}$ is cont at c.

Some useful results:
· Tut 11: $x \mapsto \sqrt{x}$ is continuous on $\mathbb{R}_0^+$
$x \mapsto \frac{1}{\sqrt{x}}$ is continuous on $\mathbb{R}^+$
$x \mapsto \frac{1}{x^2}$ is continuous on $\mathbb{R}$

CONTINUITY + TOPOLOGY

Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ s.t. f is continuous at all $c \in D$. Then, f is cont. on D.

Preservation of compactness: Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be continuous. Let $A \subseteq D$ be compact, i.e. it is closed and bounded. Then $f(A)$ is compact.

$A \subseteq \mathbb{R}$ is compact $\Leftrightarrow$ $A \subseteq \mathbb{R}$ sequentially compact.
→ note that this holds in general for $\mathbb{R}^n$ but not for other spaces e.g. metric space

Extreme Value Theorem: Let $D \subseteq \mathbb{R}$ be <u>compact</u> and let $f: D \to \mathbb{R}$ be continuous. Then f has both an absolute max and an absolute min in D.

Localization of roots: Let $a, b \in \mathbb{R}$, $a < b$ and let $f: [a, b] \to \mathbb{R}$ continuous s.t. $f(a)$ and $f(b)$ have opposite signs i.e. $f(a) > 0 \wedge f(b) < 0$ OR $f(a) < 0 \wedge f(b) > 0$. Then $\exists c \in (a, b)$ s.t. $f(c) = 0$.

Intermediate Value Theorem: Let $a, b \in \mathbb{R}$, $a < b$ and let $f: [a, b] \to \mathbb{R}$ be continuous. Let $d \in \mathbb{R}$, between $f(a)$ and $f(b)$ i.e. $f(a) < d < f(b)$ OR $f(a) > d > f(b)$. Then $\exists c \in (a, b)$ with $f(c) = d$.

Preservation of Intervals: Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be continuous. Then $f(I)$ is an interval.

Note that while continuous maps preserve intervals, they do not necessarily preserve the type of interval (i.e. its boundedness, openness etc)

UNIFORM CONTINUITY

A function $f: D \subseteq \mathbb{R} \to \mathbb{R}$ is said to be **uniformly continuous** on D if:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, u \in D: |x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$$

Sequential criterion for absence of uniform continuity: Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$. f is not unif. cont. iff $\exists \varepsilon > 0$ and a seq (x_n) in D s.t. $\lim(x_n - u_n) = 0$ AND $\forall n \in \mathbb{N}: |f(x) - f(u)| \geq \varepsilon$.

useful results:

- Lec 22: $x \mapsto x^2$ unif cont. on $[-a, a]$, $a > 0$.
  - $x \mapsto x^2$ NOT unif. cont. on $[0, \infty)$
  - $x \mapsto \frac{1}{x}$ NOT unif cont on $(0, 1]$
  - $x \mapsto \sqrt{x}$ unif cont. on $[a, \infty)$, $a > 0$.
- Lec 23: $x \mapsto \sqrt{x}$ unif cont. on $[0, \infty)$

Lemma: $\forall x, u \in \mathbb{R}$, $x \geq u \geq 0$: $\sqrt{x} - \sqrt{u} \leq \sqrt{x - u}$

Algebraic Laws for unif cont: Let f, g be unif. cont. Then:

a) $f + g$ unif cont
b) $f - g$ "
c) $\forall k \in \mathbb{R}$: $k \cdot f$ unif cont

Let $A \subseteq \mathbb{R}$ be compact. Let $f: A \to \mathbb{R}$ be cont. Then f is unif cont.

Let $A \subseteq \mathbb{R}$ be compact. Let $f, g: A \to \mathbb{R}$ be unif cont. Then $f \cdot g$ unif cont.

LIPSCHITZ CONTINUITY

Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$. f is called Lipschitz or Lipschitz cont. if $K > 0$ s.t.
$\forall x, u \in D$: $|f(x) - f(u)| < K \cdot |x-u|$.

e.g. $x \mapsto x^2$ is Lipschitz on $[-a, a]$, $a > 0$
$x \mapsto \sqrt{x}$ is Lipschitz on $[a, \infty)$, $a > 0$

Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be Lipschitz. Then f is also unif. cont (and thus continuous).

Lipschitz cont $\Rightarrow$ unif cont $\Rightarrow$ cont. But converse does not hold.