FINALS : GRAPH THEORY

DEFINITIONS AND BASICS:

$G = (V, E)$

set of nodes ↑ ↖ set of edges

- 2 nodes connected by an edge are "adjacent" : $v \sim w \Rightarrow w \sim v$
- nodes adjacent to a given node v are its "neighbours"
- "degree" : the number of vertices adjacent to a given vertex
  - ↳ complete graph : $\deg(v) = n-1$ for K_n
  - ↳ regular graph : $\deg(v)$ is constant

Theorem : HANDSHAKING LEMMA

In any graph $G = (V, E)$, the sum of the degrees of all nodes in a graph is twice the number of edges.

$$\sum_{v \in V} \deg(v) = 2|E|$$

→ Proof :

Each edge $e \in E$ is attached to 2 vertices. Thus, each edge contributes 1 to each of the vertices it connects to. Adding up the degrees of all vertices would be double-counting all the edges. Hence, $\sum \deg v = 2|E|$.

- For regular graphs : A d-regular graph with n vertices will have $\frac{dn}{2}$ edges
- implies that $\sum$ even $\deg(v)$ + $\sum$ odd $\deg(v)$ = even
  $\Rightarrow \sum$ odd $\deg(v)$ = even → there are an even number of vertices of odd degree.
- implies that a graph with an odd number of vertices has at least 1 vertex of even degree

Cauchy - Schwartz Inequality : Given $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are all $\in \mathbb{R}$

$$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leq (a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2)$$

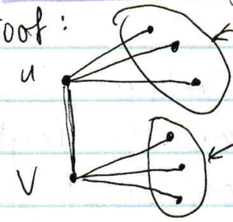
$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

$$\Rightarrow \left(\sum_{i=1}^{n} a_i\right)^2 \leq n\left(\sum_{i=1}^{n} a_i^2\right)$$

Theorem: MANTEL : $|E| \le n^2/4$

If a graph on n vertices has no triangles, then the number of edges is at most $n^2/4$.

→ Proof:



$N(u)$: neighbours of u, $N(u) \setminus \{v\}$

$N(v)$: neighbours of v, $N(v) \setminus \{u\}$

G has no triangles:

⇒ $N(u)\setminus\{v\}$ and $N(v)\setminus\{u\}$ are disjoint in $V \setminus \{u, v\}$

⇒ $\underbrace{|N(u)\setminus\{v\}|}_{\deg(u)-1} + \underbrace{|N(v)\setminus\{u\}|}_{\deg(v)-1} \le \underbrace{|V\setminus\{v,u\}|}_{n-2}$

$\deg(u) + \deg(v) \le n$

$$\sum_{u,v:\, u\sim v} (\deg(u) + \deg(v)) \le n \cdot \sum_{u,v:\, u\sim v} 1 = 2n|E|$$

↳ $$\sum_{v\in V} \deg(v)^2 = \sum_{u,v:\, u\sim v} (\deg(u) + \deg(v))$$

The squared sum of degrees of all vertices is equal to the sum of degrees of endpoints of all edges.

Using the identity,

$2 \sum \deg(u)^2 \le 2n|E|$

$\sum \deg(u)^2 \le n|E|$

Cauchy-Schwartz: $\sum \deg(u)^2 \ge \frac{1}{n} \left(\sum \deg(u)\right)^2$

$\sum \deg(u)^2 \ge \frac{1}{n} (2|E|)^2$

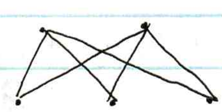
$\therefore \frac{4|E|^2}{n} \le n|E|$

$|E| \le \frac{n^2}{4}$ ▨

BIPARTITE GRAPHS:

A graph G is bipartite if its set of vertices V can be partitioned into 2 sets V_0 and V_1, so that each edge connects a vertex in V_0 to a vertex in V_1.

- complete bipartite graph: eg $K_{2,3}$:



for $K_{m,n}$:

$|V| = m+n$, $|E| = mn$

- If G is a bipartite graph with partitions V_0 and V_1,
$$\sum_{v \in V_0} \deg(v) = \sum_{u \in V_1} \deg(u) = |E|$$
  ↳ also implies that G is a regular graph of degree $d \geq 1$, $|A| = |B|$
- $K_{m,n}$ has no triangles
  - if n is even, $K_{\frac{n}{2},\frac{n}{2}}$ has n vertices, $\frac{n^2}{4}$ edges
  - if n is odd, $K_{\frac{n-1}{2},\frac{n+1}{2}}$ has n vertices, $\frac{n^2-1}{4} = \lfloor \frac{n^2}{4} \rfloor$ edges
- egs of bipartite graphs:
  - Hamming Cube : partition → $V_0 = \{x : \text{sum}(x) \equiv 0 \bmod 2\}$, $V_1 = \{x : \text{sum}(x) \equiv 1 \bmod 2\}$
  - star graph 
  - path graph 
  - cycle graphs with <u>even</u> number of vertices

WALKS, CYCLES :

"walk" : a sequence $v_1 \sim v_2 \sim v_3 \cdots \sim v_n$
  ↳ closed walk : $v_1 \sim v_2 \sim \cdots \sim v_n = v_1$ (same start and end)

"path" : a walk with distinct vertices (so paths are cases of walks)

"cycles" : a closed walk, all vertices except start/end distinct

Theorem : SHORTENING LEMMA

If there is a walk between u and v of length l, then there is a path between u and v of length $\leq l$.

→ Proof :

Let $u = v_0 \sim v_1 \sim \ldots \sim v_l = v$ be a walk from u to v
- If vertices are all distinct → it is a path
- otherwise, say $i < j$ are indices with $v_i = v_j$ (same vertex)
  - delete the closed subwalk $v_i \sim v_{i+1} \sim \ldots \sim v_j = v_i$
  - remaining walk is $v_0 \sim v_1 \sim \cdots \sim v_i \sim v_{j+1} \sim \cdots \sim v_l$ which is shorter than original, length $< l$
  - repeat until all vertices are distinct, path obtained

Proposition: On a graph G, the relation $v \approx w$: v can be joined to w by a walk is an equivalence relation.

→ Proof:

- reflexive: $\forall v \in V$, $v \approx v$, v is a walk of length 0
- symmetry: If v is joined to w by a walk, then that walk in reverse is a walk from w to v.
- transitive: if v is joined by a walk to w, and w is joined by a walk to u, then v is joined by a walk to u by concatenation.

"connectivity": A graph is connected if any 2 vertices can be joined by a path (or, equivalently, a walk)

"distance": $dist(u,v)$ is the minimum length of a path from u to v

- if graph is not connected, $dist(u,v) = \infty$

"diameter": $\max\{dist(u,v) : u, v \in V\}$

- for complete graph : diameter = 1
- cycle graph : diameter = $\frac{n}{2}$
- Hamming cube : diameter = n

Theorem: The distance on a graph satisfies:

- $dist(u,v) \geq 0$ and $dist(u,v) = 0$ iff $u = v$
- $dist(u,v) = dist(v,u)$
- $dist(u,v) \leq dist(u,w) + dist(w,v)$ ← Triangle Inequality

Lemma: if G (a connected graph) has an odd-length closed walk, then it has an odd length cycle

→ Proof:

minimality argument (contradiction)

Theorem: G is bipartite $\Leftrightarrow$ G has no odd length cycles

→ Proof:

1. Assume G is bipartite ("$\Rightarrow$")
   Every path (walk) alternates vertices in V_0 and V_1.
   Hence any cycle has even length since it must return to the vertice it started from.
2. Assume G has no odd length cycles ("$\Leftarrow$")
   Let $u \in V$ be some vertex. Partition all other vertices based on parity of distance from u:

$$X = \{v \in V : \text{dist}(u,v) \text{ even}\}$$
$$Y = \{v \in V : \text{dist}(u,v) \text{ odd}\}$$

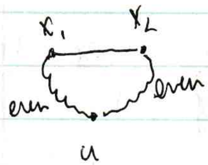
$X \cap Y = \emptyset$ since a distance cannot be both odd and even.

G is connected, so $X \cup Y = V$.

Suppose that there exists some edge incident to 2 vertices in X.

Let $x_1, x_2 \in X$, $x_1 \sim x_2$.

It follows that:



$x_1 \in X \Rightarrow \exists$ path from u to x_1, that is even in length

$x_2 \in X \Rightarrow \exists$ path from u to x_2, " "

Concatenating the edge $x_1 \sim x_2$, path from u to x_2 and path from u to x_1, we get a closed walk of odd length:

even + even + 1 = odd

By the lemma, there is an odd cycle in the graph (↯)

Hence, it must be the case that X and Y is a valid bipartition, and G is a bipartite graph.

Eulerian: G (a connected graph) is Eulerian if it has a closed walk visiting each **edge** exactly once.

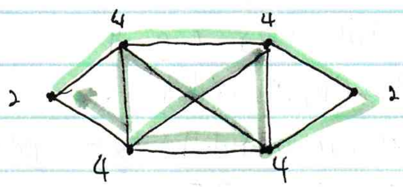
Hamiltonian: G is Hamiltonian if it has a closed walk visiting each **vertex** exactly once.

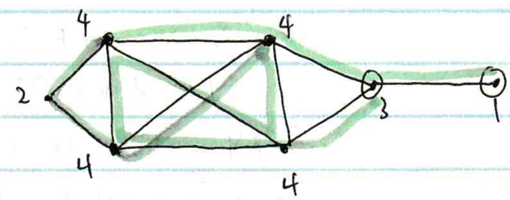
Theorem: a) If G has more than 2 vertices of odd degree, it has no Eulerian walk

b) If G has exactly 2 vertices of odd degree, it has an Eulerian walk. Every Eulerian walk must start at one of these and end at the other one.

c) If G has no vertices of odd degree, ie all of the vertices are of even degree, then it has an Eulerian walk, every Eulerian walk is closed.

eg





Only true for multigraphs (can have multiple edges for a pair of endpoints):
Every graph can be made Eulerian by adding edges.

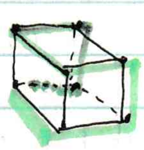
Conjecture: Every vertex-transitive graph is Hamiltonian except for small known counterexamples (eg Petersen graph)

eg cycle graph Hamiltonian
complete graph Hamiltonian

Adding edges to a Hamiltonian graph keeps it Hamiltonian.

Theorem: If a graph on n vertices has $\forall v \in V$, $\deg(v) \geq \frac{n}{2}$, then the graph is Hamiltonian. (DIRAC'S THEOREM)

eg Q_n is Hamiltonian



→ Proof: $n=2$ induction, Assume $Q_n \rightarrow Q_{n+1}$

Q_{n+1} has V_n layered into: $\{0x : x \text{ binary } n\text{-strings}\}$
$\{1x : x \text{ binary } n\text{-strings}\}$

Each layer is a copy of Q_n: $0x \sim 0y$ in Q_{n+1} if $x \sim y$ in Q_n
(joining them if they differ in 1 slot)

→ Proof of DIRAC'S THEOREM:

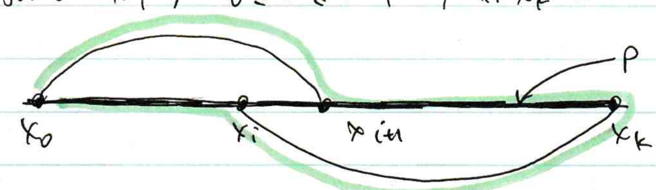
"Every graph G with $n \geq 3$ vertices and minimum degree $\deg(v) \geq \frac{n}{2}$ has a Hamiltonian cycle"

Assume a graph G with $n \geq 3$ vertices and min degree $\deg(v) \geq \frac{n}{2}$.

Then G is connected, as otherwise the degree of any vertex in a smallest component C of G would be at most $|C|-1 < \frac{n}{2}$, which contradicts the hypothesis.

Let $P = x_0 \sim x_1 \sim \cdots \sim x_k$ be the longest path in G. Since P cannot be extended to a longer path, all of x_0 and x_k's neighbours lie on P. Hence, at least $\frac{n}{2}$ of vertices $x_0, \ldots, x_{k-1}$ are adjacent to x_k, and at least $\frac{n}{2}$ of vertices $x_1, \ldots, x_k$ are adjacent to x_0. Thus, at least $\frac{n}{2}$ of the vertices $x_i \in \{x_0, \ldots, x_{k-1}\}$ are such that $x_0 \sim x_{i+1} \in E$.

Combining this statement with the pigeonhole principle, we see that there is some x_i, $0 \leq i \leq k-1$, $x_i x_k \in E$ and $x_0 x_{i+1} \in E$.



We claim the cycle $C = x_0 x_{i+1} x_{i+2} \cdots x_k x_i x_{i-1} \cdots x_0$
$= x_0 x_{i+1} P x_k x_i P x_0$
is a Hamiltonian cycle of G.

Otherwise, since G is connected, there would be some vertex x_j of C adjacent to a vertex y not in C, so that $x_j \sim y \in E$.

But then we could obtain a path longer than P ending in x_j by attaching this new edge to the k edges from x_0 to x_k.

Contradiction, as P is the longest path.

Take π: a Hamiltonian path in Q_n obtained by deleting an edge incident to 0, say x_0 is last vertex in π.
In Q_{n+1}, 0π is a path visiting each vertex $0x$ exactly once, ending at $0x_0$.
Now do $0x_0 \sim 1x_0 \ldots$, running through 1π backwards, getting to $1\ldots0\ldots0$ joining to $0\ldots.0\ldots0$.

Q_1: 0, 1

Q_2: 00, 10, 11, 01

Q_3: 000, 100, 110, 010, 011, 111, 101, 001

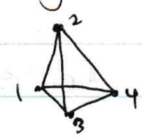
MATRICES:

- adjacency matrix: for a graph with n vertices, a 0-1 n by n matrix represents adjacency relationships.

$$A(ij) = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

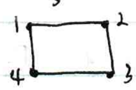
~~different adjacency matrices~~
~~give different labellings~~

eg K_4:



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$ symmetric

C_4:



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

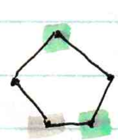
- for different labellings of vertices, there will be different but similar matrices. Eigenvalue of the matrices will be the same.
- Eigenvalues sum up to 0
- A has n real eigenvalues

GRAPH COLORING

- "k-colouring" of G is a colouring of G using k colours so that adjacent vertices have different colours

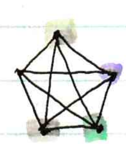
chromatic number: smallest k for which a k-colouring of G exists, $\chi(G)$

eg C_5:



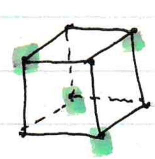
$\chi = 3$

K_5:



$\chi = 5$

Q_3:



$\chi = 2$

bipartite: $\chi = 2$
k-partite: $\chi = k$
n vertices: $\chi \leq n$

Theorem: A graph is 2-colourable $\Leftrightarrow$ it contains no odd cycles

Theorem: BROOKS' THEOREM

If each vertex degree is at most d, then the graph can be coloured with $d+1$ colours

→ Proof: by induction on vertices

Base case: $n=1$, up to $n \leq d+1$, there is a $d+1$ colouring

Inductive step: Assume known for n

Let G have $n+1$ vertices.

Omit a vertex $v \in G$ and its incident edges to obtain the remaining graph G' with n vertices.

Vertex degree still at most d. → G' can be coloured with $d+1$ colours. (inductive hypothesis)

Theorem: G is bipartite $\Leftrightarrow$ $\chi(G) = 2$

→ Proof:

"$\Rightarrow$": Assume G is bipartite

For the partition $V_0 \cup V_1 = V$, we can colour all $v \in V_0$ one colour and all $v_1 \in V_1$ another colour. There are no edges within V_0 or V_1, so this colouring is valid.

"$\Leftarrow$": Assume $\chi(G) = 2$

Colour the graph with 2 colours, red and blue.

Let V_0 be the set of red vertices, V_1 be the set of blue.

$V_0 \cap V_1 = \emptyset$, no edges within V_0 or V_1 since adjacent vertices have different colours. Hence, it is a valid bipartition.

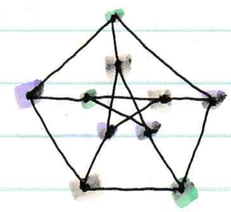
"Independence number": independent subset of G is a vertex subset where no 2 vertices are adjacent. $\alpha(G)$ is the size of the largest independence subset

Theorem: If G has n vertices, $\chi \cdot \alpha \geq n$

→ Proof:

Consider a colouring with χ colours.
For each colour, let V_k be the subset coloured by k. This forms a partition of χ sets. $\Rightarrow \sum_{k=1}^{\chi} |V_k| = n$
Each V_k is an independent subset, $|V_k| \leq \alpha$.
Hence, $n = \sum_{k=1}^{\chi} |V_k| \leq \alpha \cdot \chi$

eg: $n=10$



$\chi = 3$, $\alpha = 4$
$10 < 3 \cdot 4 = 12$

TREES:

- Defn: A tree is a connected graph with no cycles.

Theorem: A graph $G = (V, E)$ is a tree $\Leftrightarrow$ for any 2 vertices $u, v \in V$, there is a unique path joining them.

→ Proof:

"$\Rightarrow$": Assume G is a tree
Since G is connected, there is a path σ_1 connecting u to v.
Assume there is more than one path connecting u to v, say $\sigma_2 \neq \sigma_1$.
This would create a cycle in the graph, passing through u and v.
A contradiction, as trees have no cycles.
Hence, unique path from u to v.
"$\Leftarrow$": Assume unique path between any 2 vertices
G is connected, since any 2 vertices can be joined by a path.
Assume that G has a cycle.
For any 2 vertices on the cycle, there will be 2 different paths joining them, $\lightning$
Hence G has no cycles.
Since G is connected and has no cycles, it is a tree.

Proposition: A graph G is a tree if it is connected but any edge deletion disconnects G.

→ Proof:

A tree has the least number of edges among all connected graphs with the same number of vertices.

Tree-growing algorithm:
1. Start with one vertex ("seed")
2 Add a new vertex and connect to any one vertex
3. Repeat (2)

Theorem: A tree on n vertices has n-1 edges.
→ Proof by induction
→ Delete an edge, then do induction on the 2 remaining trees.

Theorem: G is a tree ⟺ G can be produced by the tree growing algorithm.
→ Proof:
"⇒" Assume G is a tree
Induction on m = # of edges
Base case: m=0 •
G is a single vertex produced by step 1.
Inductive step: Assume all trees with ≤m-1 edges are produced by the tree-growing algorithm.
Let G be a tree with m edges. Let v be a leaf in G with deg v = 1, e its edge.
Remove v and e from G to obtain G'. Then G' is still a tree and according to inductive hypothesis is produced by the algorithm. We attach a new vertex to G', joining an edge to it, producing G.
Hence G can be produced by the algorithm.

Theorem: $G=(V,E)$ is a tree ⟺ G is connected and $|E| = V-1$
→ Proof:
"⇒": G is a tree
Then G is produced by the tree growing algorithm.
At step 1, G is a single vertex. $|V|=1$, $|E| = 0 = 1-1$
At step 2, we add 1 vertex and 1 edge, hence the equality of the relation is maintained.

"$\Leftarrow$" Assume G is connected and $|E| = |V| - 1$
Assume for contradiction that G is not a tree, has a cycle σ.
Now repeat the following:

1. Remove an edge from σ. The graph will still be connected, still a walk connecting the points. The number of vertices $|V|$ does not change, but $|E|$ decreased by 1.
2. If no cycles left, done. If not, repeat.

After k steps, we will obtain a tree T with $|V|$ vertices and $|E| - k$ edges.
Contradiction as if T is a tree, then it has $|V|-1$ edges ("$\Rightarrow$")
$|E| - k = |V| - 1 = |E| \Rightarrow -k = 0$ (↯)
Hence, G must be a tree.

"pendant vertex" : a vertex in any graph of degree 1

<u>Proposition</u> : A tree on at least 2 vertices has at least 2 pendant vertices.

→ Proof 1:
Take a path of maximum length. The endpoints are pendant vertices. If not, then the path is not of maximum length.

→ Proof 2: ~~By induction~~
~~Base case: 2 vertices ... both are pendant vertices~~
Every tree has $n-1$ edges. Sum of all degrees of all vertices in the tree has to be $2(n-1)$, by the Handshaking Lemma. If there are no vertexes of degree 1, then all vertexes have at least degree 2 $\Rightarrow \sum \deg v \geq 2n$, which is a contradiction. If there is only 1 degree 1 vertex, $\sum \deg v \geq 1 + 2(n-1) > 2(n-1)$, contradiction.
Hence, there has to be at least 2 vertices of degree 1.

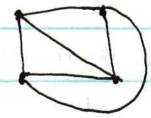
<u>Remark</u>: If a vertex in a tree has degree d, then there are at least d pendant vertices.

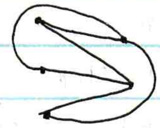
<u>Remark</u>: trees are bipartite. (no cycles)

NOT TESTED!

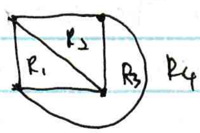
PLANAR GRAPHS :

Defn : G is planar if it can be drawn in $\mathbb{R}^2$ (ie on a plane) without any of its edges crossing each other.

eg K_4 :  is planar: 

$K_{3,2}$:  is planar : 

"faces"/"regions" : the area enclosed by the edges of a planar graph. as well as the region on the outside

eg 

THEOREM : EULER'S FORMULA

"Euler characteristic"

A planar graph satisfies $|V| - |E| + |F| = 2$

→ Proof: Induction on m = no. of edges

Base case : $m = 0$, G is connected so G is a single vertex

$|V| = 1$, $|E| = 0$, $|F| = 1$

$|V| - |E| + |F| = 2$ is true

Inductive step : Assume formula holds for any graph with $m-1$ edges. Let G have $|E| = m$.

2 cases:

1 - G is a tree.

$|F| = 1$, $|E| = |V| - 1$

$|V| - (|V| - 1) + |F| = 1 + |F|$

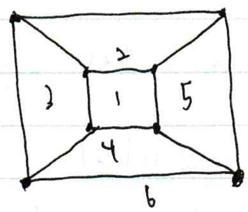
$= 2$

2 - G is not a tree, has a cycle

Let e be an edge on the cycle. On either side of e, there are distinct regions. If we remove e, graph is still connected but one less edge ⇒ one less region.

Formula applies to smaller graph:

$|V| - (|E| - 1) + (|F| - 1) = 2 = |V| - |E| + |F|$ for G eg Q_3 :



$|V| - |E| + |F|$
$= 8 - 12 + 6$
$= 2$

<u>Conjecture</u> : K_5 is not planar.

→ Proof :

Assume K_5 is planar → has Euler characteristic $= 2$

$|V| - |E| + |F| = 5 - \binom{5}{2} + |F| = 2$

$|F| = 7$

Each region has at least 3 edges on its boundary, so we must have at least $\frac{3 \cdot 7}{2} = 10.5$ edges.

Contradiction, as we have 10 edges.

Thus, K_5 cannot be planar.

<u>Remark</u> : From $|V| - |E| + |F| = 2$

$3|V| - 3|E| + 3|F| = 6$

$2|E| \geq 3|F|$

↳ since each region is bounded by at least 3 edges, $2|E|$ is the sum of all vertices' degrees.

$3|V| - 3|E| + 2|E| \geq 6$

$3|V| - |E| \geq 6$

$\Rightarrow |E| \leq 3|V| - 6$

<u>Theorem</u> : If G is a connected planar graph with no cycles of length 3 (triangles), $|E| \leq 2|V| - 4$

→ Proof :

Then each region bounded by at least 4 edges.

$2|E| \geq 4|F|$

From Euler's Formula : $8 = 4|V| - 4|E| + 4|F|$

$\leq 4|V| - 4|E| + 2|E|$

$8 \leq 4|V| - 2|E|$

$|E| \leq 2|V| - 1$

Lemma: Every planar graph has a vertex of degree ≤ 5.

→ Proof:

From Euler's formula, $|E| \leq 3|V| - 6$.

Assume for contradiction that every node in the graph has degree ≥ 6. Thus, by Handshake Lemma, $2|E| = 6|V|$,

$\Rightarrow |E| = 3|V|$, contradiction.

Theorem: FOUR COLOUR THEOREM

Every planar graph can be ~~[illegible]~~ coloured with 4 colours.

$\chi(\text{planar}) = 4$.

SPANNING TREES:

Defn: A spanning tree of a graph G is a tree which is a subgraph of G and contains all vertices of G.

Theorem: G connected $\Leftrightarrow$ G has a spanning tree

→ Proof:

"$\Rightarrow$" Assume G is connected.

Produce spanning tree T by algorithm.

1. Start with T = G
2. If T is tree, done
3. otherwise, T will be connected, not a tree hence has a cycle σ. Remove an edge of σ from T, T still connected, containing all vertices in G.

This will leave us with a spanning tree of G.

"$\Leftarrow$": G has a spanning tree

All vertices on spanning tree T are connected, T contains all vertices on G, so G is connected.