MATH223 COURSE REVIEW

VECTOR SPACE

All 10 conditions must hold for V to be a vector space:

A1. $u+v \in V$	S1. $\alpha u \in V$
A2. $u+v = v+u$	S2. $\alpha(u+v) = \alpha u + \alpha v$
A3. $(u+v)+w = u+(v+w)$	S3. $(\alpha+\beta)u = \alpha u + \beta u$
A4. $\exists\, 0_V$ s.t. $u + 0_V = u$	S4. $(\alpha\beta)u = \alpha(\beta u)$
A5. $\exists -u$ s.t $u+(-u) = 0_V$	S5. $1u = u$

Some properties:

- 0_V is unique
- $0 \cdot u = 0_V$
- $\alpha 0_V = 0_V$
- $-u = -1 \cdot u$, is unique
- $u+v = u+w \Leftrightarrow v = w$
- $\alpha u = 0_V \Leftrightarrow \alpha = 0$ or $u = 0_V$

course questions: As1 Q1

SUBSPACE

If W is a subspace of V:

1. $0_V \in W$ (W contains the zero vector)
2. $\alpha u + \beta v \in W$, $\alpha, \beta \in \mathbb{R}$, $u, v \in V$ (closed under vector addition and scalar multiplication)

in the course: As1 ~~[illegible]~~ Q2, 5, 6, 3, MIDTERM Q2

eg: As1 Q2a Is $U = \{ f \in F(\mathbb{N}) \mid \exists\, \alpha$ such that $f(k+1) = \alpha + f(k),\ k \geq 0\}$ a subspace of $F(\mathbb{N})$?

1. set $\alpha = 0$, $f(k) = 0 \ \forall k \in \mathbb{N}$.
   $f(k+1) = 0 + 0 = 0$
   $f = 0 \in U$.
2. let $f_1, f_2 \in U$, ie $f_i(k+1) = \alpha + f_i(k)$, $i = 1, 2$, $\forall k \geq 0$
   $(f_1 + f_2)(k+1) = f_1(k+1) + f_2(k+1)$
   $= (\alpha_1 + \alpha_2) + (f_1 + f_2)(k)$

$\therefore f_1 + f_2 \in U$

3. let $\beta \in \mathbb{R}$

$$(\beta f)(k+1) = \beta f(k+1)$$
$$= \beta(\alpha + f(k))$$
$$= \alpha\beta + (\beta f)(k)$$

$\therefore \beta f \in U$

Hence, U is a subspace since it contains the zero function and is closed under vector add. and scalar multiplication.

eg <u>As 1 Q5</u> Prove that $V = \{ f \in F(\mathbb{N}) \mid f(n) = f(n+3), n \geq 0 \}$ is a subspace of $F(\mathbb{N})$.

1. $f(n) = f(n+3) = 0 \quad \therefore f = 0 \in V$
2. $f, g \in V$:

$$(f+g)(n) = f(n) + g(n)$$
$$= f(n+3) + g(n+3)$$
$$= (f+g)(n+3)$$

$\Rightarrow f+g \in V$

3. $\alpha \in \mathbb{R}$:

$$(\alpha f)(n) = \alpha f(n)$$
$$= \alpha f(n+3)$$
$$= (\alpha f)(n+3)$$

$\Rightarrow \alpha f \in V$

SPANNING SETS

Vectors $\{u_1, \ldots, u_n\}$ are said to span V if every $v \in V$ is a linear combination $v = \alpha_1 u_1 + \cdots + \alpha_n u_n$ for some scalars a_i.

in the course : As 1 Q4, 5, 8, MIDTERM Q2,

eg <u>As 1 Q4</u> $D = \{1, 2, 3, 4\}$. What is the spanning set of $F(D)$?

idea: $\forall f \in F(D)$, f maps to $f(1)$, $f(2)$, $f(3)$, or $f(4) \in \mathbb{R}$.

Consider the functions :

$$f_1(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x = 2, 3, 4 \end{cases}$$

$f_2(x) = \begin{cases} 1 & \text{if } x = 2 \\ 0 & \text{if } x = 1, 3, 4 \end{cases}$

$f_3(x) = \begin{cases} 1 & \text{if } x = 3 \\ 0 & \text{if } x = 1, 2, 4 \end{cases}$

$f_4(x) = \begin{cases} 1 & \text{if } x = 4 \\ 0 & \text{if } x = 1, 2, 3 \end{cases}$

Hence, $f = f_1 f(1) + f_2 f(2) + f_3 f(3) + f_4 f(4)$

$F(D) = \text{Span}\{f_1, f_2, f_3, f_4\}$

eg <u>MIDTERM Q2b</u> $U = \{f \in P_3 \mid f(1) + f(-1) = 0\}$ is a subspace of P_2.

Find its ~~spanning set~~ basis.

$f \in U: f(x) = ax^3 + bx^2 + cx + d$, $a, b, c, d \in \mathbb{R}$.

$f(1) = a + b + c + d$

$f(-1) = -a + b - c + d$

$f(1) + f(-1) = 0 \Rightarrow b = -d$

$\therefore f(x) = ax^3 + b(x^2 - 1) + cx$

$U = \text{Span}\{x^3, x^2 - 1, x\}$

As the ~~vectors~~ span is linearly independent, that set is also U's basis.

!! Approach for identifying spans of functions:

#1: identify what f can be represented as

#2: what is the pattern? define a function to tell us which f(n) to map to

#3: sum them up; this gives you the span.

<u>LINEAR INDEPENDENCE, BASIS</u>

$S = \{u_1, \dots, u_n\}$ is linearly independent iff $\sum_{i=1}^{n} \alpha_i u_i = 0_V$, $\alpha_1 = \dots = \alpha_n = 0$.

Properties:

- $\{0_V\}$ is linearly dependent. Any set containing 0_V is linearly dependent.
- $\{u\}$ is linearly independent iff $u \neq 0_V$.

The basis is the maximal linearly independent set / the minimal spanning set.

The number of vectors in the basis is the dimension of the vector space.

Hilroy

in the course : As 1 Q8, As 2 Q1, 2, 5, MIDTERM Q2, 3b

!!! Approach for finding basis:

#1. Find spanning set

#2. Check for linear independence, adjust set accordingly.

eg MIDTERM Q3b : T / F — If $\{u, v\}$ is a linearly independent subset of a vector space V then $\dim(\text{Span}\{u+v, u-2v, 3u+v\}) = 3$.

Check if the spanning set is L.I.:

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 7/3 \\ 0 & 1 & 2/3 \end{bmatrix}$$

The spanning set is not L.I. as $3u + v = \frac{7}{3}(u+v) + \frac{2}{3}(u-2v)$.

The basis is $\{u+v, u-2v\}$, dimension of the space it spans is 2.

$\therefore$ statement is False.

eg lecture : $W = \{p \in P_3 \mid p(1) = p'(1) = p(0)\}$. Find basis and dimension of W.

recall Taylor polynomial: $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots$

$$p \in W: p(x) = p(1) + \frac{p'(1)}{1!}(x-1) + \frac{p^{(2)}(1)}{2!}(x-1)^2 + \frac{p^{(3)}(1)}{3!}(x-1)^3$$

$$= 0 + 0 + \frac{p^{(2)}(1)}{2!}(x-1)^2 + \frac{p^{(3)}(1)}{3!}(x-1)^3.$$

$W = \text{Span}\{(x-1)^2, (x-1)^3\}$

$\alpha(x-1)^2 + \beta(x-1)^3 = 0 \iff \alpha = \beta = 0$

Since spanning set is L.I. it is also a basis, $\dim(W) = 2$.

eg As 1 Q8 : $U = \{M \in M_{2\times2} \mid MJ = JM^T, \forall J \in M_{2\times2}\}$. Find basis and dimension.

for $MJ = JM^T$ to hold for every J, it is enough to show that it holds for the elementary matrices of $M_{2\times2}$.

let $J_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $J_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $J_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$MJ_1 = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$, $J_1M^T = \begin{bmatrix} a & c \\ 0 & 0 \end{bmatrix} \Rightarrow c = 0$

$MJ_2 = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$, $J_2M^T = \begin{bmatrix} b & d \\ 0 & 0 \end{bmatrix} \Rightarrow b = 0, a = d$

$\therefore U = \text{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$, $\dim(U) = 1$.

SUM AND DIRECT SUM

let E and F be 2 subspaces of V. $E+F = \{u+v \mid u \in E, v \in F\}$.
If $E \cap F = \{0_v\}$, $E \oplus F$ is the direct sum of E and F.

$E+F$ is a subspace of V.
spanning sets: $E = \text{Span}\{B\}$, $F = \text{Span}\{S\}$. For $E+F$, the spanning set is $B \cup S$. If B and S are both linearly independent sets, $B \cup S$ is linearly independent too iff $E \cap F = \{0_v\}$.

For V, a finite-dimensional vector space, $\exists$ a subspace E_1 of V such that $V = E \oplus E_1$.

If E and F are finite-dimensional subspaces of a vector space V, then
$\dim(E+F) = \dim(E) + \dim(F) - \dim(E \cap F)$.
If $E \cap F = \{0_v\}$, $\dim(E \oplus F) = \dim(E) + \dim(F)$.

in the course: As 2 Q3, 4, 6, MIDTERM Q3a, d, As 3 Q2

$\boxed{!!}$ Approach for proving sum
#1 check intersection of subspaces (direct sum?)
#2 If we can show that $\dim(E+F) = \dim(V)$, this proves $E+F = V$.

eg MIDTERM Q3a T/F — For $V = \{f \in P_2 \mid f(1) = f(-1) = 0\}$, $P_2 = V + P_1$.

$P_1 = \text{Span}\{1, x\}$, $\dim(P_1) = 2$

$f \in V$: $f(x) = ax^2 + bx + c$

$f(1) = a + b + c$

$f(-1) = a - b + c$

$f(1) = f(-1) = 0 \Rightarrow b = 0,\ a = -c \Rightarrow f(x) = a(x^2 - 1)$

$V = \text{Span}\{x^2 - 1\}$, $\dim(V) = 1$

$V \cap P_1 = \{0\}$

$\dim(P_2) = 3 = \dim(P_1) + \dim(V)$

$\therefore P_2 = V \oplus P_1$ is true.

eg <u>MIDTERM Q3d</u> T/F — E, F and G are subspaces of $\mathbb{R}^2$ such that

$$E \oplus F = E \oplus G \Rightarrow F = G.$$

consider $E = \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$, $F = \text{Span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$, $G = \text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

It is true that $E \oplus F = E \oplus G = \mathbb{R}^2$

But $F \neq G$.

Hence the statement is false.

eg <u>A1 3 Q2b</u> Let $T: V \to V$ be a linear transformation such that $T^2 - I = 0$.

Prove that $V = \ker(T - I) \oplus \ker(T + I)$

note that $T^2 - I = 0 \Rightarrow (T - I) \circ (T + I) = (T + I) \circ (T - I) = 0$

$\ker(T - I) \cap \ker(T + I) = \{0\}$:

$u \in \ker(T - I) \cap \ker(T + I) \Rightarrow (T(u) - u) = (T(u) + u) = 0$

$u = -u$ thus $u = 0$.

also $$u = \underbrace{\tfrac{1}{2}(T(u) + u)}_{\substack{\in \text{Im}(T+I) \\ \subseteq \\ \ker(T-I)}} + \underbrace{\tfrac{1}{2}(u - T(u))}_{\substack{\in \text{Im}(T-I) \\ \subseteq \\ \ker(T+I)}}$$

It follows that $\ker(T + I) \oplus \ker(T - I) = V$

<u>LINEAR TRANSFORMATION</u>

A function $T: V \to W$ is a linear transformation if $T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2)$ for $v_1, v_2 \in V$, $\alpha_1, \alpha_2 \in \mathbb{R}$.

properties:

1. $T(0_V) = 0_W$
2. $(\alpha_1 T_1 + \alpha_2 T_2)(v) = \alpha_1 T_1(v) + \alpha_2 T_2(v)$ is a linear transformation from V to W too.

Kernel of T: $\ker(T) = \{u \in V \mid T(u) = 0_W\}$

Image of T: $\text{Im}(T) = \{w \in W \mid w = T(u), u \in V\}$

In general, $\text{Im}(T) = \text{Span}\{T(u_1), T(u_2), \dots T(u_n)\}$,

u_i are the vectors in the basis of V.

!! Approach: finding kernel: what does an element in $\ker(T)$ look like?

finding image: transform basis of V, find span to get $\text{Im}(T)$.

Given $T: V \to W$, a linear transformation is said to be

- one-to-one / injective if whenever $T(u_1) = T(u_2)$, $u_1 = u_2$.
  - $Ker(T) = \{0_V\} \Leftrightarrow T$ is one-to-one
  - A L.I. subset in V transformed is still L.I. in W.
- onto / surjective if $Im(T) = W$
  - $Im(T) = Span\{T(u_i), \forall i\} = W$
- Isomorphic if T is one-to-one AND onto
  - $dim(V) = dim(W)$

In the course: As 2 Q7, MIDTERM Q1, As Q1

eg MT Q1 $T: P_3 \to \mathbb{R}^2$ such that $T(f) = \begin{bmatrix} f(1) + f(-1) \\ f(0) \end{bmatrix}$ is a linear transformation. Find a basis of $Ker(T)$ and $Im(T)$.

$Ker(T) = \{p \in P_3 \mid T(p) = \underset{\sim}{0}\}$

$p(x) = ax^3 + bx^2 + cx + d$

$T(p) = \begin{bmatrix} 2b + 2d \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow d = b = 0$

$\therefore p = ax^3 + cx \Rightarrow Ker(T) = Span\{x^3, x\}$ (basis, since L.I).

Basis of P_3: $\{1, x, x^2, x^3\}$

$T(1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$T(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = T(x^3)$

$T(x^2) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$Im(T) = Span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right\} \Rightarrow$ basis: $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right\}$

eg As 2 Q7 $T: M_{2\times2} \to P_2$, $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = tr(A)\, x^2 + c(x-1) + b$.
Find a basis of $Ker(T)$ and $Im(T)$.

$Ker(T) = \{M \in M_{2\times2} \mid T(M) = 0\}$

$M \in Ker(T)$: $(a+d)x^2 + c(x-1) + b = 0 \Rightarrow a+d = 0$, $b = c = 0$

$M = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow Ker(T) = Span\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}$

Basis of $M_{2\times2}$: $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = T(e_4) = x^2$

$T(e_2) = 1$

$T(e_3) = x-1$

$Im(T) = Span\{1, x-1, x^2\}$.

COMPOSITIONS OF LIN. TRANSF.

Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be linear transformations.
The function $T_2 \circ T_1 : V_1 \to V_3$ is also a linear transformation.

$$V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \qquad V_1 \xrightarrow{T_2 \circ T_1} V_3$$

Let $T: V \to W$ be an isomorphism. $T^{-1} : W \to V$ is also a linear transformation.

$T \circ T^{-1} = Id_W$ $\qquad W \xrightarrow{T^{-1}} V \xrightarrow{T} W$

$T^{-1} \circ T = Id_V$ $\qquad V \xrightarrow{T} W \xrightarrow{T^{-1}} V$

in the course: As 3 Q 6

eg for previous section As 3 Q1 : V is the set $\{x \in \mathbb{R} \mid x > 0\}$, equipped with $x \oplus y = xy$ and $\alpha \odot x = x^\alpha$. Find an explicit isomorphism between V and $\mathbb{R}$.

$T: V \to \mathbb{R}$, $T(x) = \ln x$, $\forall x \in \mathbb{R}$

1. T is linear:
$$\begin{aligned} T(\alpha_1 \odot x_1 \oplus \alpha_2 \odot x_2) &= T(x_1^{\alpha_1} \oplus x_2^{\alpha_2}) \\ &= T(x_1^{\alpha_1} x_2^{\alpha_2}) \\ &= \ln(x_1^{\alpha_1} x_2^{\alpha_2}) \\ &= \alpha_1 \ln(x_1) + \alpha_2 \ln(x_2) \\ &= \alpha_1 T(x_1) + \alpha_2 T(x_2) \end{aligned}$$

2. T is one-to-one and onto $\Rightarrow T$ is isomorphic.

eg As 3 Q6a T/F – If S and T are 2 isomorphisms from V into V, then $S+T$ is also an isomorphism

False. Consider $S = I$ and $T = -I$.

eg As3 Q6b T/F – V is a finite-dimensional vector space. If S and T are linear transformations from V into V s.t. $S \circ T = 0$, then $\text{rank}(T) + \text{rank}(S) \geq n$.

False. Consider $V = \mathbb{R}^2$. $S = 0$, $T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ so $\text{rank}(S) = 0$, $\text{rank}(T) = 1$.
$\text{rank}(T) + \text{rank}(S) = 1 < 2$.

RANK-NULLITY THEOREM

Let V be of finite dimension, $T: V \to W$ is a linear transformation.
Then $\dim(V) = \dim(\operatorname{Ker}(T)) + \dim(\operatorname{Im}(T))$
$= \text{nullity}([T]) + \text{rank}([T])$

in the course : As 3 Q3

eg As 3 Q3 $\dim(V) = n$, $T: V \to W$ linear, E is a subspace of V and F is a subspace of W. Let $T^{-1}(F) = \{u \in V \mid T(u) \in F\}$ and $T(E) = \{w \in W \mid w = T(u) \text{ for some } u \in E\}$.

a) Prove that $\dim(T^{-1}(F)) = \dim(\operatorname{Ker}(T)) + \dim(F \cap \operatorname{Im}(T))$

Let T_1 be the restriction of T to $T^{-1}(F)$: $\operatorname{Im}(T_1) = F \cap \operatorname{Im}(T)$
If $w \in F \cap \operatorname{Im}(T)$: $w = T(u)$ and $w \in F$.
Thus $u \in T^{-1}(F) \Rightarrow w = T_1(u)$, $w \in \operatorname{Im}(T_1)$
$\operatorname{Ker}(T_1) = \operatorname{Ker}(T)$ as $\operatorname{Ker}(T) \subseteq T^{-1}(F)$.
Applying Rank-Nullity Theorem, $\dim(T^{-1}(F)) = \dim(\operatorname{Ker}(T)) + \dim(F \cap \operatorname{Im}(T))$ ∎

b) Prove that $\dim(E) = \dim(T(E)) + \dim(\operatorname{Ker}(T) \cap E)$

Let T_2 be the restriction of T to E : $\operatorname{Im}(T_2) = T(E)$, $\operatorname{Ker}(T_2) = \operatorname{Ker}(T) \cap E$.
Hence by Rank-Nullity Theorem, $\dim(E) = \dim(T(E)) + \dim(\operatorname{Ker}(T) \cap E)$ ∎

COORDINATE REPRESENTATION

Given $B = \{u_1, \dots, u_n\}$ a basis of V, for every $u \in V$, u can be written as $u = \sum_{i=1}^{n} x_i u_i$. The vector $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is the coordinate vector of u relative to B, $[u]_B$.

in the course : As 2 Q5

eg As 2 Q5 $D = \{-2, -1, 0, 1, 2\}$. $U = \{f \in F(D) \mid f(-x) = f(x)\}$.
$S = \{g_1, g_2, g_3\}$ is a basis of U, where $g_1 = 1$, $g_2 = \frac{1}{4}x^2(1-x^2)$ and $g_3 = \frac{1}{3}x^2(4-x^2)$. Let g be a function in $F(D)$ defined as $g(2) = g(-2) = 3$, $g(1) = g(-1) = g(0) = 2$. Find $[g]_S$.

$[g]_S \in \mathbb{R}^3$, $g = ag_1 + bg_2 + cg_3$, $a, b, c \in \mathbb{R}$.
$g(0) = a\,g_1(0) + b\,g_2(0) + c\,g_3(0) = 2 \Rightarrow a = 2$
$g(1) = 2\,g_1(1) + b\,g_2(1) + c\,g_3(1) = 2 \Rightarrow c = 2$

$g(2) = 2g_1(2) + b\,g_2(2) = 3 \Rightarrow b = -\frac{1}{3}$

Hence, $[g]_S = \begin{bmatrix} 2 \\ -\frac{1}{3} \\ 0 \end{bmatrix}$

TRANSITION MATRIX

Let $B = \{u_1, \dots, u_n\}$ and $S = \{v_1, \dots, v_m\}$ be 2 bases of a vector space V. There exists a unique matrix denoted $P_{B,S}$, the transition matrix from B to S s.t. $\forall w \in V$, $[w]_S = P_{B,S}[w]_B$.

$$P_{B,S} = \left[[u_1]_S \; [u_2]_S \; \dots \; [u_n]_S \right]$$

i^{th} column of $P_{B,S}$ is the S-coordinate of the i^{th} vector in basis B.

$P_{B,S} \cdot P_{S,B} = I_n$.

$(P_{B,S}^{-1} = P_{S,B})$

MATRIX REP. OF LIN. TRANSF.

$T: V \to V$ a linear transformation, $B = \{u_1, \dots, u_n\}$ is a basis for vector space V. $\exists$ a unique $n \times n$ matrix $[T]_B$, the matrix of T relative to B such that $[T]_B [u]_B = [T(u)]_B$, $\forall u \in V$.

$$[T]_B = \left[[T(u_1)]_B \; [T(u_2)]_B \; \dots \; [T(u_n)]_B \right]$$

in the course: As 3 Q4

eg As3Q4b $T: P_2 \to P_2$, $T(p)(x) = p'(x) + xp'(x)$. $B = \{x^2 - x, x+1, x-1\}$

Find $[T]_B$.

let $p_1 = x^2 - x$, $p_2 = x+1$, $p_3 = x-1$

$T(p_1)(x) = (2x-1) + x(2x-1) = 2x^2 + x - 1 = 2p_1 + p_2 + 2p_3$

$T(p_2)(x) = 1 + x = p_2$

$T(p_3)(x) = 1 + x = p_2$

Hence $[T]_B = \left[[T(p_1)]_B \; [T(p_2)]_B \; [T(p_3)]_B \right] = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$

DIAGONALIZATION OF $[T]_B$

Given $T: V \to V$, B and S are 2 bases of V, $[T]_B$ and $[T]_S$ are similar. ie $[T]_B$ is diagonalizable and $[T]_S$ is diagonal.

$$[T]_B = P_{S,B} [T]_S P_{S,B}^{-1}$$

!! Approach for understanding the question: "Is there a basis S s.t. $[T]_S$ is diagonal?" $\equiv$ "Is $[T]_B$ diagonalizable?"

in the course: As 3 Q 2c, 4, 5

eg As3Q5 $B = \{u, v, w\}$ is a basis of a vector space V. $T: V \to V$ s.t. $T(u) = v + w$, $T(w) = u + v$, $T(v) = u + w$. Find a basis S s.t. $[T]_S$ is diagonal.

$$[T]_B = [[T(u)]_B \ [T(v)]_B \ [T(w)]_B]$$
$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\det([T]_B - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}\right)$$
$$= -(\lambda+1)^2(\lambda-2) = 0$$

$\lambda_1 = 2, \quad \lambda_2 = \lambda_3 = -1$

eigenvectors are: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore [T]_B = PDP^{-1}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

As $[T]_B$ and $[T]_S$ are similar, ie $[T]_B = P_{S,B} [T]_S P_{S,B}^{-1}$,
$D = [T]_S$, $P = P_{S,B}$

Recall that $P_{S,B} = [[s_1]_B \ [s_2]_B \ [s_3]_B]$, s_i are vectors in basis S.

Hence, $S = \{u+v+w, \ -u+v, \ -u+w\}$.

$T: V \to V$ is an isomorphism iff $[T]_B$ is invertible.

$[T^{-1}]_B = ([T]_B)^{-1}$.

MATRIX REP. OF GENERAL LIN. TRANSF.

$T: V \to W$, B is a basis of V, $B = \{u_1, \dots, u_n\}$ and S is a basis of W, $S = \{q_1, \dots, q_m\}$. The vectors $T(u_1), T(u_2), \dots, T(u_n)$ are in W, and hence can be represented as a linear combination of q_i.

$[T]_{B,S}$ is the matrix representation of T relative to B and S, such that $[T(u)]_S = [T]_{B,S}[u]_B \quad \forall u \in V$

$$[T]_{B,S} = \Big[\, \underbrace{[T(u_1)]_S}_{S \text{ coordinates of } T(u_i)} \;\; [T(u_2)]_S \;\; \dots \;\; [T(u_n)]_S \,\Big]$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "i_B"'] & W \arrow[d, "i_S"] \\ \mathbb{R}^n \arrow[r, "\tilde{T}"'] & \mathbb{R}^m \end{tikzcd}$$

coord. map in B — i_B; i_S — coord map in S

$$(\tilde{T} \circ i_B)(u) = (i_S \circ T)(u)$$
$$[T]_{B,S}[u]_B = [T(u)]_S$$

in the course: As 4 Q2

eg <u>As 4 Q2</u> let $B = \{x^2, x, 1\}$ and $S = \{x^2+x, x-1, x+1\}$ be 2 bases of P_2. $T: P_2 \to P_2$ such that $[T]_{B,S} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$.

a) Prove that $\ker(T) = i_B^{-1}(\text{null}([T]_{B,S}))$

let $\underset{\sim}{x} \in \text{null}([T]_{B,S}) \iff [T]_{B,S}\,\underset{\sim}{x} = \underset{\sim}{0}$

$$[T]_{B,S}[i_B^{-1}(\underset{\sim}{x})]_B = \underset{\sim}{0}$$
$$[T(i_B^{-1}(\underset{\sim}{x}))]_S = \underset{\sim}{0}$$
$$T(i_B^{-1}(\underset{\sim}{x})) = 0 \quad (\text{in } P_2)$$
$$\therefore\; i_B^{-1}(\underset{\sim}{x}) \in \ker(T)$$

Thus, $i_B^{-1}(\text{null}([T]_{B,S})) = \ker(T)$. ∎

b) Prove that $\text{Im}(T) = i_S^{-1}(\text{col}([T]_{B,S}))$

let $v \in \text{Im}(T) \iff v = T(u)$, ie $[v]_S = [T]_{B,S}[u]_B$

$$[v]_S = i_S(v) \in \text{col}([T]_{B,S})$$

$\Rightarrow v \in i_S^{-1}(\mathrm{col}([T]_{B,S}))$

Thus, $\mathrm{Im}(T) = i_S^{-1}(\mathrm{col}([T]_{B,S}))$ ∎

eg lecture $V = P_2$, $W = M_{2\times 2}$, $T: V \to W$ such that $T(p)(x) = \begin{bmatrix} a+b & b \\ c & a \end{bmatrix}$ for $p(x) = ax^2 + bx + c$. Using the canonical bases, find $[T]_{B,S}$.

$[T]_{B,S} = \big[\, [T(u_1)]_S \;\; [T(u_2)]_S \;\; [T(u_3)]_S \,\big]$, u_i - i vector in basis of V.

$B = \{1, x, x^2\}$

$$\left.\begin{aligned} T(1) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ T(x) &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ T(x^2) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}\right\} \text{Hence, } [T]_{B,S} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

eg lecture $T: P_3 \to P_2$, $B = \{x^3 + 2x,\ x^2 + 2,\ 1 + x,\ 1 - x\}$ basis of P_3 and $S = \{x^2, x, 1\}$ basis of P_2. $[T]_{B,S} = \begin{bmatrix} 1 & 2 & -1 & 6 \\ 2 & -1 & 3 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix}$. Find Ker(T) and Im(T).

note that $\mathrm{Ker}(T) = i_B^{-1}(\mathrm{null}([T]_{B,S}))$, $\mathrm{Im}(T) = i_S^{-1}(\mathrm{col}([T]_{B,S}))$.

$$\begin{bmatrix} 1 & 2 & -1 & 6 \\ 2 & -1 & 3 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$a = -c - 2d$, $b = c - 2d$

$\mathrm{null}([T]_{B,S}) = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\mathrm{Ker}(T) = \mathrm{Span}\left\{ i_B^{-1}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right), i_B^{-1}\left(\begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}\right) \right\}$

$= \mathrm{Span}\{ -x^3 + x^2 - x + 3,\ -2x^3 - 2x^2 - 5x - 3 \}$

$\mathrm{col}([T]_{B,S}) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$

$\mathrm{Im}(T) = \mathrm{Span}\left\{ i_S^{-1}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right), i_S^{-1}\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\right) \right\}$

$= \mathrm{Span}\{ x^2 + 2x + 1,\ 2x^2 - x \}$

INNER PRODUCT

The map that assigns a real number to a pair of vectors, $u, v \in V$, is an inner product $\langle u, v \rangle$ if the following conditions hold:

1. linearity : $\langle \alpha_1 u_1 + \alpha_2 u_2, v \rangle = \alpha_1 \langle u_1, v \rangle + \alpha_2 \langle u_2, v \rangle$
2. symmetric : $\langle u, v \rangle = \langle v, u \rangle$
3. positive definite : $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \Leftrightarrow u = 0$.

let A be a symmetric matrix. A is said to be positive definite if:
1. $\forall\, u \in \mathbb{R}^n$, $u^T A u \geq 0$
2. $u^T A u = 0 \iff u = 0$.

In general, A is positive definite if its eigenvalues are positive and it is a symmetric matrix.

in the course: A1 4 Q1

eg A1 4Q1 let $\langle u, v\rangle = u^T A v$. Is $\langle u, v\rangle$ an inner product for the following As?

a) $A = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$

no, as A is not symmetric, so $\langle u, v\rangle \neq \langle v, u\rangle$

b) $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

A must be positive definite: $\langle u, u\rangle = u^T A u \geq 0$.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}\right)$$
$$= (1-\lambda)^2 - 4 = 0$$

$\lambda_1 = -1$, $\lambda_2 = 3 \Rightarrow$ not positive definite, not inner product.

c) $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$

A is both symmetric and positive definite, hence $\langle u, v\rangle$ is an inner prod.

ORTHOGONAL VECTORS

The norm of $u \in V$, V is an inner product space, is denoted
$\|u\| = \sqrt{\langle u, u\rangle}$

orthogonality: 2 vectors, $u, v \in V$, are said to be orthogonal when $\langle u, v\rangle = 0$.

properties:
- Cauchy-Schwarz inequality: $|\langle u, v\rangle| \leq \sqrt{\langle u, u\rangle}\sqrt{\langle v, v\rangle}$
- Pythagorean equality: if $\langle u, v\rangle = 0$, $\|u + v\|^2 = \|u\|^2 + \|v\|^2$

ORTHOGONAL BASIS

Some definitions:

- an orthogonal set is a set wherein any 2 elements are orthogonal to each other. $S = \{u_1, \dots, u_n\}$, $\langle u_i, u_j \rangle = 0$, $i \neq j$
- proper orthogonal set: orthogonal set excluding 0_V
- orthonormal set: normalized orthogonal set, $\|u_i\| = 1 \ \forall i$

Every finite (proper) orthogonal set is linearly independent.

An orthogonal basis of V is the orthogonal set with n vectors, $\dim(V) = n$.
Let $B = \{u_1, \dots, u_n\}$ be an orthogonal basis of V. Every $v \in V$ can be written as: $v = \sum_{i=1}^{n} \frac{\langle v, u_i \rangle}{\langle u_i, u_i \rangle} u_i$

When B is orthonormal, this simplifies to $v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i$

eg lecture $V = P_1$, $\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$. Find an orthogonal basis of V, then an orthonormal basis.

set $p_1(x) = 1$. want to find $p_2(x)$ s.t. $\langle p_1, p_2 \rangle = 0$
let $p_2(x) = ax + b$. $\Rightarrow \langle p_1, p_2 \rangle = \int_{-1}^{1} ax + b \, dx = 2b \Rightarrow b = 0$
$p_2(x) = x$
$B = \{1, x\}$ is an orthogonal basis of V.
$\|p_1\| = \sqrt{\langle p_1, p_1 \rangle} = \sqrt{2}$
$\|p_2\| = \sqrt{\langle p_2, p_2 \rangle} = \sqrt{\frac{2}{3}}$
Hence, the orthonormal basis of V is $\{ 1/\sqrt{2}, \sqrt{3/2}\, x \}$

eg A1 4 Q4 V is finite-dimensional inner product space. $T: V \to \mathbb{R}$. Show that there exists a unique vector u_0 such that $T(v) = \langle u_0, v \rangle$, $\forall v \in V$.

Let $B = \{u_1, u_2, \dots, u_n\}$ be an orthonormal basis of B.
Hence $v \in V$ can be expressed as $v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i$.
$T(v) = T(\sum_{i=1}^{n} \langle v, u_i \rangle u_i)$
$= \sum_{i=1}^{n} T(u_i) \langle u_i, v \rangle$
$= \langle \sum_{i=1}^{n} T(u_i) u_i, v \rangle$
Thus, $u_0 = \sum_{i=1}^{n} T(u_i) u_i$

Hilroy

Thus u_0 is unique:

Assume that $\langle v, u_1 \rangle = \langle v, u_2 \rangle = T(v)$, $\forall v \in V$

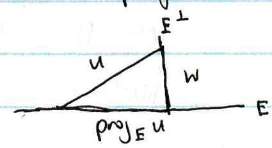
$\langle v, u_1 \rangle - \langle v, u_2 \rangle = 0 \Rightarrow \langle v, u_1 - u_2 \rangle = 0$

$u_1 - u_2 = 0 \Rightarrow u_1 = u_2$.

ORTHOGONAL PROJECTION

Let V be a vector space equipped with an inner product. E is a subspace of V, $\dim(E) = n$, $B = \{u_1, \ldots, u_n\}$ is an orthogonal basis of E.
For all $u \in V$, there is a unique vector p in E such that the vector $w = u - p$ is orthogonal to every vector in E. p is the projection of u onto E:

$$\mathrm{proj}_E\, u = \sum_{i=1}^{n} \frac{\langle u, u_i \rangle}{\langle u_i, u_i \rangle} u_i$$



If B is an orthonormal basis, $\mathrm{proj}_E\, u = \sum_{i=1}^{n} \langle u, u_i \rangle u_i$

Some properties:

1. Bessel inequality: $\| u \|^2 \geq \sum_{i=1}^{n} \frac{\langle u, u_i \rangle^2}{\langle u_i, u_i \rangle}$, $\forall u \in V$
2. Parseval inequality: $\| u \|^2 = \sum_{i=1}^{n} \frac{\langle u, u_i \rangle^2}{\langle u_i, u_i \rangle}$, $\forall u \in E$.

eg As 4 Q3 $D = \{-1, 0, 1\}$. The inner product on $F(D)$ is defined as
$\langle f, g \rangle = f(-1)\, g(-1) + 2\, f(0)\, g(0) + 4\, f(1)\, g(1)$.
E is the subspace of all even functions defined on D.
$g(x) = x^2 + x$, $\forall x \in D$. Find $\mathrm{proj}_E\, g$.

note: even functions: $f(1) = f(-1)$

Basis of $E = \{f_1, f_2\}$, where $f_1 = \begin{cases} 1 & \text{if } x = 1 \text{ or } -1 \\ 0 & \text{if } x = 0 \end{cases}$

and $f_2 = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \text{ or } -1 \end{cases}$

$\langle f_1, f_2 \rangle = 0$, so this is an orthogonal basis.

$$\mathrm{proj}_E\, g = \sum_{i=1}^{2} \frac{\langle f_i, g \rangle}{\langle f_i, f_i \rangle} f_i = \frac{\langle f_1, g \rangle}{\langle f_1, f_1 \rangle} f_1 + \frac{\langle f_2, g \rangle}{\langle f_2, f_2 \rangle} f_2$$

$$= \frac{8}{5} f_1 + 0 f_2$$

$$= \frac{8}{5} f_1$$

GRAM-SCHMIDT ORTHOGONALIZATION PROCESS

Suppose $B = \{u_1, u_2, \dots, u_n\}$ is a basis of V, an inner product space. With the Gram-Schmidt algorithm, we can use B to construct an orthogonal basis $\{v_1, v_2, \dots, v_n\}$ of V:

1. set $v_1 = u_1$
2. $$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$$

$\vdots$

$$v_n = u_n - \frac{\langle u_n, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_n, v_2\rangle}{\langle v_2, v_2\rangle} v_2 - \dots - \frac{\langle u_n, v_{n-1}\rangle}{\langle v_{n-1}, v_{n-1}\rangle} v_{n-1}$$

In general, $v_k = u_k - \text{proj}_{E_{k-1}} u_k$

eg lecture $V = P_2$, $\langle f, g\rangle = \int_{-1}^{1} f(t)\,g(t)\,dt$. $B = \{1, x, x^2\}$ is a basis of P_2. Find an orthogonal basis for V.

Let orthogonal basis be $\{q_1, q_2, q_3\}$.

set $q_1 = 1$

$$q_2 = x - \frac{\langle p_2, q_1\rangle}{\langle q_1, q_1\rangle} q_1$$

$$= x$$

$$q_3 = x^2 - \frac{\langle p_3, q_1\rangle}{\langle q_1, q_1\rangle} q_1 - \frac{\langle p_3, q_2\rangle}{\langle q_2, q_2\rangle} q_2$$

$$= x^2 - \frac{1}{3}$$

orthogonal basis $= \{1, x, x^2 - \frac{1}{3}\}$

$\|q_1\| = \sqrt{2}$

$\|q_2\| = \sqrt{\frac{2}{3}}$

$\|q_3\| = \sqrt{\frac{8}{45}}$

orthonormal basis $= \{1/\sqrt{2}, \sqrt{3/2}\,x, \sqrt{45/8}\,(x^2 - \frac{1}{3})\}$

ORTHOGONAL DIAGONALIZATION

Given an $n \times n$ symmetric A, $A = PDP^T$, $P^T = P^{-1}$.

If λ_1 and λ_2 are distinct eigenvalues of A, X_1 and X_2, the corresponding eigenvectors are orthogonal ie $X_1^T X_2 = 0$.

!! **Approach for orthogonal diagonalization**

1. diagonalize A as usual: $A = QDQ^{-1}$
2. use columns of Q as ~~[illegible]~~ vectors in a basis
3. apply Gram-Schmidt to find the orthogonal basis, ~~which make up the columns of P such that $A = PDP^T$~~ and then normalize it
4. the orthonormal basis makes up the columns of P such that $A = PDP^T$.

eg A14 Q5 $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. Find P and D for $A = PDP^T$.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{bmatrix}\right)$$
$$= (1-\lambda)^3 - 12(1-\lambda) + 16 = 0$$

$\lambda = 5$ or $\lambda = -1$

eigenvectors: $\lambda = 5$: $\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = -1$: $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$A = QDQ^{-1}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

let u_i = i^{th} column in Q:

$$p_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$p_2 = u_2 - \frac{p_1 \cdot u_2}{p_1 \cdot p_1} p_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$p_3 = u_3 - \frac{p_1 \cdot u_3}{p_1 \cdot p_1} p_1 - \frac{p_2 \cdot u_3}{p_2 \cdot p_2} p_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}$$

$\|p_1\| = \sqrt{3}$
$\|p_2\| = \sqrt{2}$
$\|p_3\| = \sqrt{\frac{3}{2}}$

Hence, $A = PDP^T$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$$

ORTHOGONAL COMPLEMENTS

Let S be a subset of V, an inner product space. The orthogonal set to S is denoted $S^\perp = \{ v \in V \mid \langle u, v \rangle = 0, \forall u \in S \}$

$S^\perp$ is a subspace of V.

Assuming S is finite, $S^\perp = (\text{Span}\{S\})^\perp$

For E, a subspace of V with finite dimension, $E \oplus E^\perp = V$, $E^\perp$ is the orthogonal complement of E.

In most cases, $(E^\perp)^\perp = E$.

eg lecture $V = P_2$, $\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$, $E = \{ f \in P_2 \mid f(1) = f(-1) = 0 \}$.
Find a basis of $E^\perp$.

$E = \text{Span}\{x^2 - 1\}$

$E^\perp = \{ p \in P_2 \mid \langle p, f \rangle = 0 \}$, $f = x^2 - 1$

$p(x) = ax^2 + bx + c \Rightarrow \langle p, f \rangle = \int_{-1}^{1} (ax^2 + bx + c)(x^2 - 1)\, dx = 0$

$\Rightarrow a + 5c = 0$

$p(x) = (-5c)x^2 + bx + c = c(1 - 5x^2) + bx$

Hence, $E^\perp = \text{Span}\{1 - 5x^2, x\}$

eg lecture $V = M_{2\times2}$, $\langle A, B \rangle = tr(AB^T)$. $E = \{ M \in V \mid M = M^T \}$. Hence,
$E = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$. Find $E^\perp$'s basis.

$E^\perp = \{ M \in V \mid tr(MA_1^T) = tr(MA_2^T) = tr(MA_3^T) = 0 \}$

let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $tr(MA_1^T) = 0 \Rightarrow a = 0$

$tr(MA_2^T) = 0 \Rightarrow d = 0$

$tr(MA_3^T) = 0 \Rightarrow b = -c$

Hence, $E^\perp = \text{Span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$