ECON 452 : ADV. MICRO THEORY

TOPICS :

· COALITIONAL GAMES	Lec#1 - 3
· NASH BARGAINING SOL$^{\underline{N}}$	4 - 5
· MATCHING	6 - 9
· MECHANISM DESIGN	10 - 11

- Bayesian
- VCG

$\Big\}$ 1st price, 2nd price auctions, public good, trade

COALITIONAL GAMES

game : (N, V)

$\nwarrow$ $V(S)$: payoffs

assumptions / conditions :

- $v(N) \geq \sum_{k=1}^{K} v(S_k)$ ∀ partitions $S_k \in N$.
- $v(S \cup T) \geq v(S) + v(T)$ $\forall S, T \subset N$ s.t. $S \cap T = \emptyset$.

$$\text{core}(N, V) = \{ x \in \mathbb{R}^n \mid \overset{\text{1. feasible}}{\sum_{i \in N} x_i \leq \underbrace{v(N)}_{\text{payoff of grand coalition}}} \text{ and } \overset{\text{2. not blocked.}}{\sum_{i \in S} x_i \geq v(S) \ \forall S \subseteq N} \}$$

axioms:

1. symmetry : if i and j are interchangeable then they contribute the same amount to every coalition of other agents
   i, j interchangeable $\Rightarrow \psi_i(N, v) = \psi_j(N, v)$
2. dummy player : dummy players should receive a payment equal to exactly the amount they achieve on their own.
   i dummy $\Rightarrow \psi_i(N, v) = v(\{i\})$
3. additivity : for any 2 v_1, v_2, we have for any player i that $\psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi_i(N, v_2)$ where the game $(N, v_1 + v_2)$ is defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ $\forall S \subseteq N$.

⇒ Shapley value satisfies these axioms + efficiency.

↖ average marginal contribution of i over all different sequences that could build up the grand coalition

$$\psi_i(N, v) = \frac{1}{n!} \underset{\text{all orderings}}{\sum_{R \in \mathcal{R}}} \Delta_i(\underset{\text{i's predecessors in ordering R.}}{S_i(R)})$$

$$\Delta_i(S_i(R)) = v(S_i(R) \cup \{i\}) - v(S_i(R))$$

$\sum \psi_i = v(N)$: i.e. sum of marginal contributions is grand coalition's wo

$$\psi_i(N, v) = \sum_{S \subseteq N} \frac{(s-1)!\,(n-s)!}{n!} [\underbrace{v(S) - v(S \setminus \{i\})}_{\text{i's marginal contribution to S.}}]$$

$$\frac{\underbrace{(s-1)!}_{\text{arrangement of i's predecessors.}} \underbrace{(n-s)!}_{\text{i's successors}}}{\underset{\text{total ways of arranging}}{n!}} \Bigg\} \text{ probability that } i \text{ enters the coalition with exactly the agents } S \setminus \{i\}.$$

NBS

NASH BARGAINING SOLUTION

bargaining problem : (F, v)

set of feasible payoffs ↗ ↖ disagreement point
(rational, i.e. > disagreement)

solution : $\Phi(F, v) \in F$.

axioms: 1. Pareto optimality
2. Individual rationality : $\Phi_i(F, v) \geq v_i \ \forall i$
3. scale invariance / independent of expected utility representation
4. independence of irrelevant alternatives

solving for NBS:
1. identify disagreement point
2. Pareto optimality / efficiency condition?
3. solve FOC for maximization problem:

$$\Phi(F, v) = \underset{x \in F,\ x \geq v}{\arg\max} \ (x_1 - v_1)(x_2 - v_2)$$

feel free to substitute in numbers!

MATCHING

- $j \succsim_i i \Rightarrow j$ is acceptable to i (better than being alone)
- matching μ is individually rational if $\mu(i)$ acceptable $\forall i$
- matching μ is blocked by pair $(f, w) \in F \times W$ if $w \succ_f \mu(f)$ and $f \succ_w \mu(w)$.
- stable matching : 1. individually rational, 2. no blocking pair

- Gale-Shapley algo results in a stable matching.

The core = set of stable matchings.

When preferences are strict, $\exists$ an F-optimal stable matching and a W-optimal stable matching.
→ proof on next page.

MATCHING

Let $w \in W$ be achievable for $f \in F$ if $\exists$ a stable matching wherein w and f are matched, i.e. $\exists\ \mu$ stable s.t. $\mu(f)=w$

First, we show that f will never be rejected by achievable w on day 1. We show this by contradiction.
Assume that w has rejected f. This only occurs when w has received a better offer, say f', i.e. $f' \succ_w \mu(w) = f$.
Since f' is proposing, this also means that $w \succ_{f'} \mu(f')$, otherwise f' would not propose. Hence, (f', w) form a blocking pair to μ, which is a contradiction since we defined that μ is stable. Hence, our assumption that f can be rejected by achievable w is wrong.

We assume that there have been no rejections up till day k. It follows that on day k, there ~~so~~ should also be no rejection:
Once again, by contradiction, assume that f has been rejected by $\mu(f) = w$. Let f' be w's best offer thus far, i.e. $f' \succ_w f$.
It must be that this is f''s first offer, since we assume that there have been no rejections thus far. Hence f''s preferred partner is w, and they form a blocking pair, contradicting the stability of μ.

As such, since firms will never be rejected by their achievable workers, it will result in a F-optimal stable matching (where every firm likes the matching at least as much as any other stable matching).

Similarly, we apply the same reasoning to conclude that workers will also have a W-optimal stable matching under the deferred acceptance algorithm when they propose first.

MATCHING

When preferences are strict, the common preferences of the 2 sides of the market are opposed on the set of stable matchings. If μ and μ' are 2 stable matchings, then all firms like μ at least as much as μ' iff all workers like μ' at least as much as μ.

→ Proof: $(\mu \succsim_f \mu' \;\; \forall f \iff \mu' \succsim_w \mu \;\; \forall w)$.

1. Show $\mu \sim_f \mu' \;\; \forall f \iff \mu'^{*} \sim_w \mu \;\; \forall w$.

   : $\mu(i) \sim \mu'(i) \;\; \forall \; i \in F \cup W \Rightarrow \mu(i) = \mu'(i)$ since preferences are strict. In both matchings, i has the same partner. Thus it follows that when $\mu(f) = w = \mu'(f)$ and $\mu(w) = f = \mu'(w)$, f and w are indifferent between μ and μ'.

2. Now consider strict preferences. We exclude firms and workers that have the same partner in both matchings. Let F_1, W_1 be the set of firms, workers respectively that have the same partner in μ and μ'.

"$\Rightarrow$": $\mu(f) \succ_f \mu'(f) \;\; \forall f \in F \setminus F_1 \Rightarrow \mu'(w) \succ_w \mu(w) \;\; \forall w \setminus W_1$.

We know that $\mu(f) \neq f$. μ' is stable, so $\mu'(f) \succsim_f f$ by individual rationality. Since μ is strictly preferable to f, then $\mu(f) \succ_f f$. By contradiction, consider that for some $w \in W \setminus W_1$, $\mu(w) \succ_w \mu'(w)$. By the same logic as before, $\mu(w) \neq w$. $\Rightarrow \mu(w) = f$, some $f \in F \setminus F_1$. So now we have $\mu(w) = f$, $\succ_w \mu'(w)$ and $\mu(f) = w \succ_f \mu'(f)$. As such, (f, w) form a blocking pair against μ', contradicting the fact that μ' is stable. Hence, the assumption that $\mu(w) \succ_w \mu'(w)$ is wrong.

"$\Leftarrow$": same idea.

MATCHING

Rural Hospital Theorem: The set of all agents who are matched in a given market is the same for all stable matchings. (given strict prefs)

→ Proof #1:

Let $\mu(F)$ be the set of firms matched under μ, and $\mu(W)$ be the corresponding set of workers, μ stable.

Let μ^F be the firm-optimal stable matching.

$|\mu^F(W)| \geq |\mu(W)|$ since any firm ~~worker~~ matched in μ would also be matched in μ^F. If this inequality does not hold, then $\exists f \in \mu(W)$ such that $f \notin \mu^F(W)$. $\Rightarrow$ under μ^F, f is alone i.e. $\mu^F(f) = f$.

However, $\mu(f) \succsim_f f$ (individual rationality) and $\mu^F(f) \succ_f \mu(f)$.

Also, $\mu(f) \neq f$, f is matched to some worker in μ.

Thus, we have a contradiction: $\mu^F(f) = f \succ_f \mu(f) \neq f$, as μ is stable. (if f prefers being alone, then it blocks μ).

Similarly $|\mu(W)| \geq |\mu^W(W)|$.

By the same logic $|\mu^W(F)| \geq |\mu(F)| \geq |\mu^F(F)|$.

Note that $|\mu^W(F)| = |\mu^W(W)|$ and $|\mu^F(F)| = |\mu^F(W)|$ since this is a one-to-one matching.

In particular, we have $|\mu^F(W)| \geq |\mu(W)| = |\mu(F)| \geq |\mu^F(F)|$ which are equivalent $\Rightarrow |\mu^F(W)| = |\mu(W)|$.

Since any ~~man~~ worker matched under μ is also matched in μ^F, it follows that $\mu^F(W) = \mu(W)$.

The same argument applies to $\mu^W(F) = \mu(F)$.

→ Proof #2.

Let F, W be finite sets, μ and μ' be stable matchings.

1. If $\mu(i) \sim_i \mu'(i)$, then because of strict preferences, $\mu(i) = \mu'(i)$. Let $j = \mu(i) = \mu'(i)$. It also holds that $\mu(j) = \mu'(j) = i$. Hence if $i \in F \cup W$ is indifferent across μ and μ', i is matched in ~~the~~ both matchings to the same partner.

2. Now consider case where an agent prefers one stable matching to another. WLOG, consider f_1 s.t. $\mu'(f_1) \succ_{f_1} \mu(f_1)$. Let $w_1 = \mu'(f_1)$, we know that $w_1 \neq f$ (not alone).

~~It follows that $\mu^*(w_1) \neq f_1$, or else that would be a blocking pair.~~ Then correspondingly, $\mu'(w_1) = f_1$. However, it cannot be that $\mu'(w_1) = f_1 \succ_{w_1} \mu(w_1)$, since this would mean that (f_1, w_1) is a blocking pair to μ. Because preferences are strict, it must be that w_1 prefers some $f_2 \neq \mu'(w_1) = f_1$, i.e. $f_2 \succ_{w_1} \mu'(w_1) = f_1$. Applying the same logic, we find that a cycle will be created, where agents' preferred partners have already been matched under another stable matching. As there is a finite number of agents, this will be a closed cycle.

Therefore, the set of agents that are matched is always the same across all matchings.

No stable matching mechanism exists for which stating the true preferences is a dominant strategy for every agent (someone will want to lie!)

↳ with a mechanism yielding the F-optimal stable matching, it is a dominant strategy ~~for~~ for firms to state true prefs. (same for W-optimal and workers).

MECHANISM DESIGN

Idea: A principal faces multiple agents who hold private information. The principal would like to condition their actions on this info.
Given the principal's objective, is it possible (if so, how?) to decentralize the decision power among individual agents in such a way that by freely exercising this decision power, agents eventually select the very outcomes that the principal considers a priori desirable?

direct mechanism: principal asks agents for their private information directly
→ how much do you value the object?
indirect mechanism: principal deduces the truth by asking indirect questions
→ how much would you bid for the object?

Overview:

- **VICKREY - CLARKE - GROOVES (VCG) MECHANISM**
  - direct mechanism, agents report their utility functions over the set of alternatives, mechanism designer selects a solution that maximizes total utility (utilitarian) wrt the reported utility function.
  - agents receive a payment from the mechanism designer equal to the externality that they impose on the other agents.

  Key results:
  - reporting truthfully is a weakly dominant strateg
  - VCG mechanism can result in a budget deficit, e.g. in public good provision, bilateral trade problems

- **BAYESIAN MECHANISM**
  - mechanism induces a static game of incomplete / private information among agents
  - defn of Bayesian Nash eq:
    - a profile of strategies (one for each player) such that each agent's strategy is a best response to other player's strategies in equilibrium
    - a profile of strategies such that for each agent and for every type this agent has, their strategy specifies an action that maximizes their conditional exp. utility.

  Key results:
  - Revelation Principle: feasibility of Bayesian Nash eq $\Rightarrow \exists$ incentive compatible direct mechanism to achieve the allocation and payments in the eq.

VCG MECHANISM

Set-up:

- $N = \{1, 2, \cdots, n\}$ agents + 1 "principal"
- A = set of alternatives, agents need to jointly choose an $a \in A$.
- each agent has a $v_i : A \to \mathbb{R}$, valuation for each alternative $a \in A$, but this is private info
- Given that $a \in A$ is selected, and the principal pays agents a payment profile $t = (t_1, \cdots, t_n)$, agents gain quasi-linear utility $v_i(a) + t_i$, where t_i can be positive or negative
- principal seeks a utilitarian a^* to maximize total utility of all agents

The mechanism

1. Agents report their utility functions to the principal: $u = (u_1, \cdots, u_n)$
2. Principal chooses $x = (a^*, t)$ as a function of u, $a^* \in \operatorname{argmax}_{b \in A} \sum_{i \in N} v_i(b)$
3. Principal makes payments to agents based on the externality that each agent imposes on all other agents:

$$t_i(u) = \underbrace{\sum_{j \neq i} u_j(a^*(u))}_{\substack{\text{total utility of all other}\\ \text{agents, when i is present}}} - \underbrace{\sum_{j \neq i} u_j(a^*(u_{-i}))}_{\substack{\text{total utility of all other agents}\\ \text{when i is not present}}}$$

- if $a^*(u) = a^*(u_{-i})$, i has no externality, $t_i(u) = 0$
- if $a^*(u) \neq a^*(u_{-i})$, i is pivotal, $t_i(u)$ is positive / negative if i has pos / neg ext.

Truth-telling is weakly dominant under VCG:

Consider utility of agent i of type v_i reporting u_i: $v_i(a^*(u)) + t_i(u)$

We can partition $u = (u_i, u_{-i})$: $v_i(a^*(u_i, u_{-i})) + t_i(u_i, u_{-i})$

$$= v_i(a^*(u_i, u_{-i})) + \sum_{j \neq i} u_j(a^*(u_i, u_{-i})) - \sum_{j \neq i} u_j(a^*(u_{-i}))$$

Consider the first 2 terms, since last one does not depend on u_i ⇒ treat as constant

If i is truthful, then $u_i = v_i$:

$$v_i(a^*(v_i, u_{-i})) + \sum_{j \neq i} u_j(a^*(v_i, u_{-i})) = \text{i's actual utility} + \text{all other agent's actual utility}$$

$$= \text{total utility for all agents}$$

Total utility cannot be less than any other solution (upper bound!), i.e. $v_i(a^*(v_i, u_{-i})) + \sum_{j \neq i} u_j(a^*(v_i, u_{-i})) \geq v_i(b) + \sum_{j \neq i} u_j(b) \quad \forall b \in A.$

Therefore, truth-telling is a weakly dominant strategy. ■

2nd price sealed bid auction:

- Set-up: $N = \{1, \cdots, n\}$ buyers, 1 seller with a single object
- $a^*(u)$: allocate object to highest bidder,
  $a^*(u_{-i})$: allocate object to highest bidder in $N \setminus \{i\}$
- payments:
  - i highest bidder, k highest bidder in $N \setminus \{i\}$ ⇒ $t_i(u) = (0 + \overset{\substack{\text{i wins object,}\\ \text{everyone gets 0}}}{\overset{\downarrow}{\cdots}} + 0) - (\overset{\substack{\text{without i, k will}\\ \text{win the auction}}}{\overset{\downarrow}{u_k}} + 0 + \cdots + 0) = -u_k$
  - $\ell \neq i$ is the highest bidder ⇒ $t_i(u) = (u_\ell + \underset{\ell \text{ wins object}}{\underset{\uparrow}{0}} + \cdots + 0) - (u_\ell + \underset{\ell \in N \setminus \{i\}\text{, wins}}{\underset{\uparrow}{0}} + \cdots + 0) = 0$

MECHANISM DESIGN

bilateral trade

- Set-up: 1 buyer, 1 seller, u_b and u_s are private
- $a^*(u) = \begin{cases} \text{trade if } u_b \geq u_s \\ \text{X trade if } u_b < u_s \end{cases}$
- Payments:
  - under trade, $t_b = 0 - \overset{\text{keeps}}{u_s} = -u_s$
    $t_s = u_b - 0 = u_b$
  - under Xtrade, $t_b = u_s - u_s = 0$
    $t_s = 0 - 0 = 0$

public good provision:

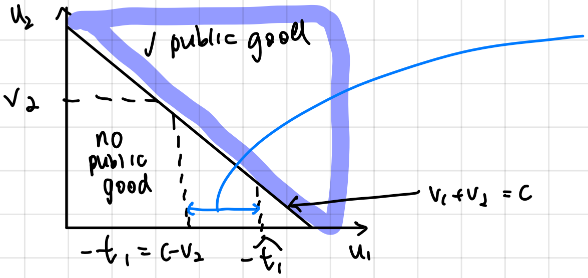
- Set-up: 2 agents with private valuations for a discrete public good, $v_i \sim \text{uniform}(0, 1)$, cost of providing the good > 1 (so single agent cannot provide), say $c = \frac{6}{5}$

VCG mechanism is as follows:

- add a 3rd player ("seller" of the public good) to the game, $t_3 = v_3 = c = \frac{6}{5}$
- agents 1 and 2 report their valuation of the public good, $\hat{v}_i$
- decision rule (following utilitarian solution) and payments from mechanism designer to agents:
  - if $\hat{v}_1 + \hat{v}_2 \geq c$, provide public good
    $t_i = \hat{v}_j - c$, $i = 1, 2$
    $t_3 = c$
    In total, payments $= t_1 + t_2 + t_3$
    $= \hat{v}_2 - c + \hat{v}_1 - c + c$
    $= \hat{v}_1 + \hat{v}_2 - c \geq 0$, implying possibility of budget deficit.
  - if $\hat{v}_1 + \hat{v}_2 < c$, no provision
    $t_i = 0$, $i = 1, 2, 3$

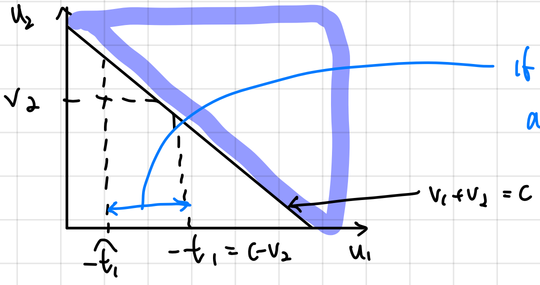
Can we solve the budget deficit problem by changing payment amounts?

- What happens if we offer a modified payment $-\hat{t}$ greater than VCG payment t?



If agent's true value lies here, then net utility after offered payment $-\hat{t}$ will be negative, so agent 1 will deviate by lying that v_1 is lower

- Similarly, if we offer modified payment less than VCG payment: $-t_1 > -\hat{t}_1$



If v_1 lies here, payment $\hat{t}_1$ results in less utility. agent can deviate by lying that their v_1 is higher

Thus, we dont have a way of resolving the budget deficit. (Note that VCG does not always result in budget deficit, e.g. 1st and 2nd price auction)

We can calculate agents' conditional exp. utility :

Given that truth-telling is weakly dominant, public good provided $\Leftrightarrow v_i + v_j \geq \frac{6}{5}$

$v_i, v_j \in [0, 1] \Rightarrow$ we need $v_i, v_j \geq \frac{6}{5} - 1 = \frac{1}{5}$ ← as no single agent can obtain the public good.

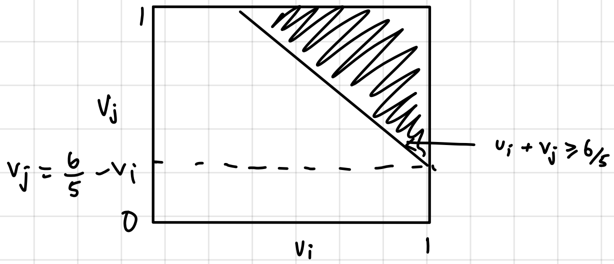
For $v_i \in [\frac{1}{5}, 1]$, i gets $v_i + v_j - \frac{6}{5}$.

i's cond. exp. utility :

$$\int_{\frac{6}{5} - v_i}^{1} (v_i + v_j - \tfrac{6}{5}) \; f(v_j) \; dv_j$$

$$= \int_{\frac{6}{5} - v_i}^{1} (v_i + v_j - \tfrac{6}{5}) \; dv_j$$

$$= \frac{1}{50} (5v_i - 1)^2$$



Integrate over v_i :

$$\int_{\frac{1}{5}}^{1} \frac{1}{50} (5v_i - 1)^2 \; f(v_i) \; dv_i$$

$$= \int_{\frac{1}{5}}^{1} \frac{1}{50} (5v_i - 1)^2 \; dv_i$$

$$= \frac{32}{375}$$

We will see more of this method in the Bayesian mechanisms section 🤗

BAYESIAN MECHANISM

Revelation Principle: Take any mechanism. If a Bayesian Nash equilibrium outcome, i.e. allocation and payments, is feasible, then there exists a direct mechanism with a truth-telling equilibrium that we can implement to achieve the same outcome.

↖ "incentive compatible direct mechanism"

Similarly, a direct mechanism is Bayesian incentive compatible if reporting true valuations by all agents is a Bayesian Nash equilibrium.

This result is useful because mechanism designers can just focus on incentive compatible direct mechanisms to design an optimal mechanism.

A quick review of games w asymmetric info

e.g. #1 market entry:

- 2 firms, an incumbent (firm 1) and potential entrant (firm 2)
- firm 1 is deciding whether to build a new plant; simultaneously firm 2 is deciding whether to enter the market.
- firm 1's cost of building is private, firm 2 only knows it is 1.5 with prob = $\frac{1}{3}$ and 0 with prob = $\frac{2}{3}$.

$p = \frac{1}{3}$, high cost

	enter	X enter
build	0, -1	$2, 0^*$
X build	$2^*, 1^*$	$3^*, 0$

$p = \frac{2}{3}$, high cost

	enter	X enter
build	1.5, -1	$3.5^*, 0^*$
X build	$2^*, 1^*$	3, 0

Notice that firm 2's payoffs are the same regardless of the building cost, but firm 2's decision on whether to enter or not changes.

e.g. #2 Cournot duopoly

- market demand: $P = a - Q$
- Firm 1: $C_1(q_1) = cq_1$, Firm 2: $C_2(q_2) = \begin{cases} c^H q_2 & \text{with prob } \theta \\ c^L q_2 & \text{" } 1-\theta \end{cases}$

firm 2's cost function is privately known, but firm 1 has incomplete info.

Harsanyi's static Bayesian games:

$$G = (N, \{A_i\}_{i\in N}, \{T_i\}_{i\in N}, \{p_i\}_{i\in N}, \{u_i\}_{i\in N})$$

set of players → N

actions for each player → $\{A_i\}_{i\in N}$

each player may be one of several types → $\{T_i\}_{i\in N}$

uncertainty → $\{p_i\}_{i\in N}$

utility function depending on $a_i \ \forall i$, t_i → $\{u_i\}_{i\in N}$

- uncertainty is specified by $p_i(t_{-i} \mid t_i)$, derived from a commonly known prior distribution $p \in \Delta(T)$

$$p_i(t_{-i} \mid t_i) = \frac{p(t_i, t_{-i})}{\sum_{t_{-i} \in T_{-i}} p(t_i, t_{-i})}$$

- types are assigned to each i by "nature", based on p → i chooses $a_i \in A_i$ → receives a payoff $u_i(a_i, a_{-i}; t_i)$

MECHANISM DESIGN

We can further simplify the game by considering strategies instead of actions and types:
- strategies $s_i : T_i \to A_i$, $s_i \in S_i$
- expected utility : $u_i(s_1, s_2, \dots, s_n) = \sum_{t \in T} p(t)\, u_i(s_1(t_1), \dots, s_n(t_n) ; t_i)$

$s \in S$ is a Bayesian Nash eq of G iff s is Nash eq of $G' = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$
i.e. for every agent $i \in N$, their strategy s_i maximizes their conditional expected utility

$$s_i \in \underset{s_i \in S_i}{\text{argmax}}\ u_i(s_i, s_{-i})$$

Back to e.g. #1 market entry:
- firm 1's strategy : $s_1 : T_1 \to A_1 = S_1 : \{H, L\} \to \{\text{build}, X \text{ build}\}$, 4 strategies
- firm 2 : $S_2 : \{\text{enter}, X \text{ enter}\}$ 2 strategies
- We can find the expected utilities for each firm for each of the $2 \times 4 = 8$ strategy outcomes

firm 1		firm 2	
H	L	enter	X enter
b	b	$\frac{2}{3}(1.5), -1$	⋮
b	Xb	$\frac{1}{3}(2), \frac{-1}{3} + \frac{2}{3}$	
Xb	b	⋮	
Xb	Xb		

=

firm 1		firm 2	
H	L	enter	X enter
b	b	$1, -1$	$3, 0^*$
b	Xb	$\frac{4}{3}, \frac{1}{3}^*$	$\frac{8}{3}, 0$
Xb	b	$\frac{5}{3}, -\frac{1}{3}$	$\underline{\frac{10}{3}^*, 0^*}$
Xb	Xb	$\underline{2^*, 1^*}$	$3, 0$

Bayesian Nash eq are ((Xb, b), (X enter)) and ((Xb, Xb), (enter)).

BNE for 1st price sealed bid auction:

Set-up:
- 2 bidders, $N = \{1, 2\}$
- each bidder i can submit a non-negative bid $A_i = [0, \infty)$, which is a function of their "type", their valuation / willingness to pay $v_i \sim \text{uniform}[0,1]$.
- highest bidder wins the auction, and pays their bid
- if tie, each wins with equal probability 0.5

Let the bidding strategy be linear, $b_i(v_i) = \alpha + \beta v_i$, $\beta > 0$
There are 2 constraints : $b_i(v_i) \geq 0$ (non-negative bid) and $b_i(v_i) \leq v_i$ (indiv. rationality)
$\Rightarrow \alpha = 0 \Rightarrow b_i(v_i) = \beta v_i$

We want to find i's conditional expected utility.
Assume that other agents have the same bidding strategy, i.e. it is symmetric.
There are 3 possible outcomes when i bids x :

1. x is the highest bid, so i wins the auction and gains net utility of $(v_i - x)$
   $\text{Prob}(x > \beta v_{-i}) = \text{Prob}(v_{-i} < \frac{x}{\beta})$
   $= \frac{x}{\beta}$ (because of uniform dist assumption)
2. x is not the highest, so i gets 0.
3. tie, win $v_i - x$ with probability $\frac{1}{2}$, $u_i = \frac{1}{2}(v_i - x)$
   But $\text{prob}(x = \beta v_j) = 0$, due to uniform dist.

cond. exp. utility:

First, integrate over the possible values of v_j, $v_j \in [0, \frac{x}{\beta}]$ for i to win

$$u_i(x, \beta v_j \mid v_i) = \int_0^{x/\beta} (v_i - x)\ f(v_j)\ dv_j$$

$$= (v_i - x)\left(\frac{x}{\beta}\right) \equiv \text{net utility} \cdot \Pr(i \text{ wins})$$

i wants to submit a bid that maximizes this utility, i.e. $\max_x (v_i - x)\left(\frac{x}{\beta}\right)$:

FOC wrt x: $-\frac{x}{\beta} + \frac{1}{\beta}(v_i - x) = 0$

$$\Rightarrow v_i - 2x = 0$$

$$\Rightarrow x = \frac{1}{2} v_i$$

Hence, for a 2-player 1st price sealed bid auction, the eq. strategy is $b_i(v_i) = \frac{1}{2} v_i$.

Furthermore, truth-telling is BNE here. Consider i's cond. exp. utility when they have type v_i, but report w_i instead, assuming j is truthful:

$$u_i(w_i, v_j \mid v_i) = (v_i - \tfrac{1}{2} w_i) \cdot \text{Prob}(i \text{ wins})$$

$$= (v_i - \tfrac{1}{2} w_i) \cdot \text{Prob}(w_i > v_j)$$

$$= (v_i - \tfrac{1}{2} w_i) \cdot w_i$$

FOC wrt w_i: $v_i - w_i = 0$

$\Rightarrow v_i = w_i$, telling the truth is optimal. $\blacksquare$

In general, for a 1st price auction with $N = \{1, \cdots, n\}$ players, the bidding strategy is given by $b_i(v_i) = \frac{n-1}{n} v_i$:

cond. exp. utility when i is of type v_i, but reports w_i, and all other agents tell the truth:

$$u_i(w_i, v_{-i} \mid v_i) = (v_i - \beta w_i) \cdot \text{Prob}(i \text{ wins})$$

$$= (v_i - \beta w_i) \cdot \text{Prob}(w_i > v_j \ \forall j \neq i)$$

$$= (v_i - \beta w_i) \cdot w_i^{n-1}$$

FOC wrt w_i: $-\beta w_i^{n-1} + (n-1)(v_i - \beta w_i) w_i^{n-2} = 0$

$$\Rightarrow w_i^{n-2}(-\beta w_i + n v_i - v_i - n\beta w_i + \beta w_i) = 0$$

$$\Rightarrow w_i^{n-2}(n v_i - v_i - n\beta w_i) = 0$$

In BNE, truth-telling can maximize the cond. exp. utility, i.e. $u_i(v_i, v_{-i})$ max.

$\Rightarrow$ FOC holds for $w_i = v_i$:

$$v_i^{n-2}(n v_i - v_i - n\beta v_i) = 0$$

$$\Rightarrow n v_i^{n-1} - v_i^{n-1} - n\beta v_i^{n-1} = 0$$

$$\Rightarrow n v_i - v_i - n\beta v_i = 0$$

$$\Rightarrow \beta v_i = \frac{n-1}{n} v_i \quad \blacksquare$$

MECHANISM DESIGN

Calculating expected revenue in 2-player 1st price auction:

- seller collects $\frac{1}{2} v_i$ from winner i if $v_i > v_j$

$$\mathbb{E}R = \frac{1}{2} v_i \cdot \text{Prob}(v_i > v_j)$$
$$= \frac{1}{2} v_i^2$$

integrate over possible values of v_i: $\int_0^1 \frac{1}{2} v_i^2 \; f(v_i)\, dv_i = \frac{1}{2}\left[\frac{1}{3} v_i^3\right]_0^1 = \frac{1}{6}$

in total (multiply by n): $\frac{1}{6} \times 2 = \frac{1}{3}$

For the general 1st price auction, total expected revenue is:

$$R_{FSA} = n \int_0^1 \frac{n-1}{n} v_i \quad Pr(v_i > v_j, \forall j \neq i) \quad \underset{\overset{\shortparallel}{1}}{f(v_i)}\, dv_i$$
$$= n \int_0^1 \frac{n-1}{n} v_i \; (v_i^{\,n-1})\, dv_i$$
$$= (n-1) \int_0^1 v_i^{\,n}\, dv_i$$
$$= n-1 \left[\frac{v_i^{\,n+1}}{n+1}\right]_0^1$$
$$= \frac{n-1}{n+1}$$

Another way to think of this is: In the 1st price auction, the seller's expected revenue is the winner's bid, i.e. expected highest bid. Given the assumption of uniform dist, the expected valuations of agents is known. For uniform $(0,1)$, the i^{th}-highest valuation is $\frac{n-i+1}{n+1}$. Thus, the expected highest valuation is $\frac{n}{n+1}$.

$$R_{FSA} = \mathbb{E}\text{ highest } b_i(v_i) = \frac{n-1}{n}\,\mathbb{E}\text{ highest } v_i = \frac{n-1}{n} \cdot \frac{n}{n+1} = \frac{n-1}{n+1} \quad ✓$$

BNE for 2nd price sealed bid auction:

Set up:

- same as 1st price auction, each player submits a bid, has valuation $v_i \sim \text{unif}[0,1]$
- highest bid wins, but pays 2nd-highest bid

Key result: $b_i(v_i) = v_i$ (i.e. bid your true valuation) is a weakly dominant strategy.

Firstly, bidding truthfully weakly dominates overbidding:

1. i wins the auction: If i tells the truth, $b_i = v_i > b_{-i}$. If i overbids $\hat{b}_i > v_i$, $\hat{b}_i > b_{-i}$. Regardless, i still pays the second-highest bid i.e. $b' = \max\{b_{-i}\}$ and gains $(v_i - b')$.
2. i loses: if i tells the truth, $b_i = v_i < b_j$ for some $j \neq i$. Let the winning bid be b^*. If i overbids $\hat{b}_i$, there are 2 outcomes: If $\hat{b}_i < b^*$, i still loses, so the utility is still 0. But if $\hat{b}_i > b^* > v_i$, then i wins the auction but will gain negative utility, $v_i - b^* < 0$.

Next, bidding truthfully also weakly dominates underbidding:

1. i loses: If i tells the truth $b_i = v_i < b_j$ for some $j \neq i$. If i underbids $\hat{b}_i < v_i < b_j$, i still loses the auction. In both cases, i gets the same utility of 0.
2. i wins: If i tells the truth, i gains utility $v_i - b'$, where b' is the second-highest bid. If i underbids but still wins, i.e. $v_i > \hat{b}_i > b'$, then i still wins the auction and pays b'. However, if i underbids and loses, i.e. $v_i > b' > \hat{b}_i$, then i gets utility 0 which is worse off. ▨

expected revenue in 2-player 2^{nd} price auction:

- seller collects v_j from i, if $v_i > v_j$ (i.e. i wins)

$$\mathbb{E}R = v_j \cdot \text{Prob}(v_i > v_j)$$
$$= v_j \cdot (1 - v_j)$$

integrate over possible values of v_j : $\int_0^1 v_j (1 - v_j) \, f(v_j) \, dv_j$

$$= \int_0^1 [v_j - v_j^2] \, dv_j$$
$$= \left[\tfrac{1}{2} v_j^2 - \tfrac{1}{3} v_j^3 \right]_0^1$$
$$= \tfrac{1}{2} - \tfrac{1}{3}$$
$$= \tfrac{1}{6}$$

in total (multiply by n) : $2 \times \frac{1}{6} = \frac{1}{3}$

For the general, n-player game:

$$R_{SSA} = n \int_0^1 (n-1) \, v_i \cdot \Pr(v_i \text{ 2nd greatest}) \; f(v_i) \, dv_i$$

$$= n \int_0^1 (n-1) \, v_i \; \underset{\substack{\uparrow \\ \text{one } v > v_i}}{(1 - F(v_i))} \; \underset{\substack{\uparrow \\ n-2 \; v_s < v_i}}{(F^{n-2}(v_i))} \; f(v_i) \, dv_i$$

$$= n \int_0^1 (n-1) \, v_i (1 - v_i)(v_i^{n-2}) \, dv_i$$
$$= n(n-1) \int_0^1 v_i^{n-1} - v_i \; dv_i$$
$$= n(n-1) \left[\frac{v_i^n}{n} - \frac{v_i^{n+1}}{n+1} \right]_0^1$$
$$= n(n-1) \left(\frac{1}{n} - \frac{1}{n+1} \right)$$
$$= n(n-1) \cdot \frac{1}{n(n+1)}$$
$$= \frac{n-1}{n+1}$$

Revenue Equivalence Theorem: If there are 2 bidders with values drawn from Unif$[0,1]$, then any standard auction has an expected revenue of $\frac{1}{3}$ and gives a bidder with value v and expected profit of $\frac{1}{2} v^2$, the same as the 2^{nd} price auction.

MECHANISM DESIGN

Provision of public good:

Set-up:

- $N = \{1, 2\}$ agents with private independent values $v_i \sim Unif(0,1)$ for a discrete public good
- cost of provision $c = \frac{3}{2}$

Generally, the mechanism is that each agent makes a contribution (not direct mech), and if the total contribution $\geq$ cost of provision, the good will be provided.

There are typically 3 games which can occur:

1. contribution game: contribution not refunded if no provision, principal keeps budget surplus
2. subscription game: contribution refunded if no provision, principal keeps budget surplus
3. optimal mechanism: contribution refunded if no provision, budget surplus is shared equally

BNE in contribution game:

Only 1 equilibrium, nobody will contribute and probability of public good provision is 0.
i.e. $b_i(v_i) = 0 \quad \forall i$

cond. exp utility for i: $v_i \cdot Prob(b_i + b_j \geq c) - b_i \leq v_i \cdot Prob(b_i + v_j \geq c) - b_i$
since $b_j \leq v_j$ due to rationality

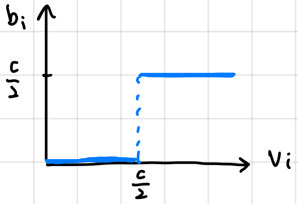
$$\begin{aligned} v_i \cdot Prob(b_i + v_j \leq c) - b_i &= v_i \cdot Prob(v_j \geq c - b_i) - b_i \\ &= v_i(1 - c + b_i) - b_i \qquad \text{) unif. dist.} \\ &= v_i - cv_i + v_ib_i - b_i \\ &= -\tfrac{1}{2}v_i + v_ib_i - b_i \qquad \downarrow \ c = \tfrac{3}{2} \\ &= \underbrace{(v_i - 1)b_i}_{<0} \ \underbrace{-\tfrac{1}{2}v_i}_{<0} \quad \Big\} < 0 \end{aligned}$$

$\Rightarrow$ i's exp. cond. utility < 0 if $b_i > 0$. Thus the only BNE that exists is 0-contribution, since a positive contribution makes agents strictly worse off.

BNE in subscription game:

1. 0-contribution: $b_i(v_i) = 0 \ \forall i$, $Prob(provision) = 0$
2. threshold strategy (splitting cost):

$$b_i(v_i) = \begin{cases} 0 & \text{if } v_i < \frac{c}{2} \\ \frac{c}{2} & \text{if } v_i \geq \frac{c}{2} \end{cases}$$



cond. exp. utility for i:

#1. $v_i < \frac{c}{2} \Rightarrow b_i(v_i) = 0$, 0-contribution eq ✓

#2. $\frac{c}{2} \leq v_i \leq 1 \Rightarrow b_i(v_i) = \frac{c}{2}$

cond. exp. utility: $\underbrace{(v_i - \frac{c}{2})}_{\geq 0} \underbrace{Prob(v_j \geq \frac{c}{2})}_{>0} > 0$

$Prob(provision) = Prob(v_i \geq \frac{c}{2}) \cdot Prob(v_j \geq \frac{c}{2}) = (1 - \frac{c}{2})^2$

i's cond. exp. utility (over v_j): $\int_{\frac{c}{2}}^{1} (v_i - \frac{c}{2}) f(v_j) \, dv_j = (v_i - \frac{c}{2})(1 - \frac{c}{2})$

i's exp utility over v_i: $\int_{\frac{c}{2}}^{1} (v_i - \frac{c}{2})(1 - \frac{c}{2}) f(v_i) \, dv_i = (1 - \frac{c}{2})^2 \cdot \frac{1}{2}(1 - \frac{c}{2})$

$\equiv Prob(provision) \cdot$ net utility

3. linear strategy : $b_i(v_i) = \alpha + \beta v_i$ (symmetry $\Rightarrow$ α, β same $\forall i$.)

Specifically, $b_i(v_i) = \frac{1}{2} v_i + \frac{1}{6}(2c-1)$:

let i's bid be x. We seek to maximize i's EU :

$$EU_i(x, b_j(v_j) \mid v_i) = (v_i - x) \cdot \text{Prob(provision)}$$
$$= (v_i - x) \cdot \text{Prob}(x + b_j(v_j) \geqslant c)$$

Thus we have the constraint $x + b_j(v_j) \geqslant c$:

$\Rightarrow x + \alpha + \beta v_j \geqslant c$

$\Rightarrow v_j \geqslant \frac{1}{\beta}(c - x - \alpha)$

v_j is bounded : $\frac{1}{\beta}(c-x-\alpha) \leqslant v_j \leqslant 1$

$\therefore EU_i = (v_i - x) \cdot \text{Prob}(v_j \geq \frac{1}{\beta}(c-x-\alpha))$

$= (v_i - x)(1 - \frac{1}{\beta}(c-x-\alpha))$

FOC wrt x : $(v_i - x)(\frac{1}{\beta}) - (1 - \frac{1}{\beta}(c-x-\alpha)) = 0$

$\Rightarrow \frac{1}{\beta}(v_i - x) - 1 + \frac{1}{\beta}(c-x-\alpha) = 0$

$\Rightarrow v_i - x - \beta + c - x - \alpha = 0$

$\Rightarrow x = \frac{1}{2} v_i + \frac{1}{2}(c - \alpha - \beta)$

$x = \alpha + \beta v_i \Rightarrow \beta = \frac{1}{2}$, $\alpha = \frac{1}{2}(c - \alpha - \beta)$

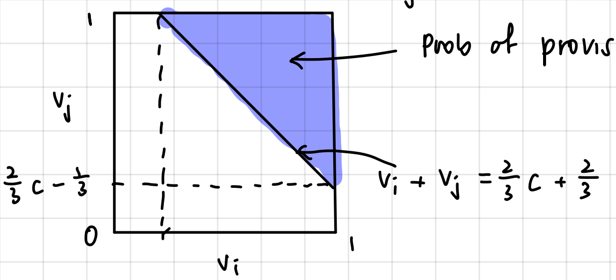
$= \frac{1}{2}c - \frac{1}{2}\alpha - \frac{1}{2}$

$\Rightarrow \alpha = \frac{1}{3}c - \frac{1}{6}$

Hence $b_i(v_i) = \frac{1}{2} v_i + \frac{1}{3}c - \frac{1}{6} = \frac{1}{2} v_i + \frac{1}{6}(2c-1)$. ■

$$\text{Prob(provision)} = \text{Prob}(\underbrace{b_i(v_i) + b_j(v_j) \geqslant c}_{\frac{1}{2}v_i + \frac{1}{6}(2c-1) + \frac{1}{2}v_j + \frac{1}{6}(2c-1) \geqslant c})$$

$\Rightarrow v_i + v_j \geqslant \frac{2}{3}c + \frac{2}{3}$



Prob of provision = area of this triangle

$= \frac{1}{2}(1 - (\frac{2}{3}c - \frac{1}{3}))^2$

$= \frac{1}{2}(\frac{4}{3} - \frac{2}{3}c)^2$

$= \frac{1}{2}(\frac{4}{3}(1 - \frac{c}{2}))^2$

$= \frac{8}{9}(1 - \frac{c}{2})^2$

each agent's expected utility : $\frac{8}{9}(1-\frac{c}{2})^2 (\frac{4}{9}(1-\frac{c}{2}))$

Comparison of equilibriums :

	EU	
threshold strategy :	$(1-\frac{c}{2})^2 \cdot \frac{1}{2}(1-\frac{c}{2})$	← better off.
linear strategy :	$\frac{8}{9}(1-\frac{c}{2})^2 \cdot \frac{4}{9}(1-\frac{c}{2})$	

MECHANISM DESIGN

BNE in optimal mechanism:

1. 0-contribution
2. threshold strategy
   - same as subscription mechanism! $\mathbb{E}U_i = \frac{1}{2}(1-\frac{c}{2})^3$
   - dont need to worry about splitting budget surplus since there's none, $\Sigma b_i = c$ exactly
3. linear strategy:

$$b_i(v_i) = \frac{2}{3} v_i + \frac{1}{4} c - \frac{1}{6} \quad \forall i$$

cond. exp. utility :

for values of v_j where public good is provided,

$$\mathbb{E}U_i(x, b_j(v_j) \mid v_i) = \int_?^? v_i - x + \underbrace{\frac{1}{2}(x + b_j(v_j) - c)}_{\text{split budget surplus}} f(v_j)\, dv_j$$

constraint is $x + b_j(v_j) \geq c$

$$\Rightarrow x + \alpha + \beta v_j \geq c$$

$$\Rightarrow v_j \geq \frac{1}{\beta}(c - x - \alpha)$$

bounds of v_j gives us the range of integration:

$$\int_{\frac{1}{\beta}(c-x-\alpha)}^{1} v_i - x + \frac{1}{2}(x + \alpha + \beta v_j - c)\, f(v_j)\, dv_j$$

$$= (v_i - \frac{x}{2} - \frac{c}{2} + \frac{\alpha}{2})(1 - \frac{1}{\beta}(c - x - \alpha)) + \frac{\beta}{2}\int_{\frac{1}{\beta}(c-x-\alpha)}^{1} v_j\, dv_j$$

$$= (v_i - \frac{x}{2} - \frac{c}{2} + \frac{\alpha}{2})(1 - \frac{1}{\beta}(c - x - \alpha)) + \frac{\beta}{2}(\frac{1}{2})(1 - \frac{1}{\beta}(c - x - \alpha))^2$$

FOC wrt x: $-\frac{1}{2}(1 - \frac{1}{\beta}(c-x-\alpha)) + (v_i - \frac{x}{2} - \frac{c}{2} + \frac{\alpha}{2})(\frac{1}{\beta}) + \frac{\beta}{4}(\frac{2}{\beta})(c - x - \alpha)(\frac{1}{\beta}) = 0$

$$\Rightarrow 2v_i - x - c + \alpha - \beta + c - x - \alpha + c - x - \alpha = 0$$

$$\Rightarrow x = \frac{2}{3} v_i + \frac{1}{3}(c - \alpha - \beta)$$

$x = b_i(v_i) = \alpha + \beta v_i \Rightarrow \alpha = \frac{1}{3}(c - \alpha - \beta), \quad \beta = \frac{2}{3}$

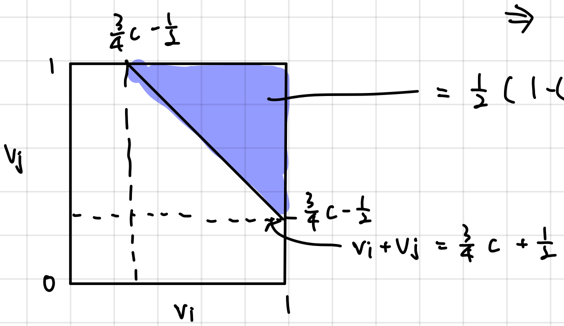
$$\Rightarrow \alpha = \frac{1}{3}c - \frac{1}{3}\alpha - \frac{2}{9}$$

$$\Rightarrow \alpha = \frac{c}{4} - \frac{1}{6}$$

$\therefore\ b_i(v_i) = \frac{2}{3} v_i + \frac{c}{4} - \frac{1}{6}$ ■

Prob(provision): $b_i(v_i) + b_j(v_j) \geq c \Rightarrow \frac{2}{3}(v_i + v_j) + 2(\frac{c}{4} - \frac{1}{6}) \geq c$

$$\Rightarrow v_i + v_j \geq \frac{3}{4}c + \frac{1}{2}$$



$$= \frac{1}{2}(1 - (\frac{3}{4}c - \frac{1}{2}))^2 = \frac{1}{2}(\frac{3}{2} - \frac{3}{4}c)^2$$

$$= \frac{1}{2}(\frac{3}{2})^2(1 - \frac{c}{2})^2$$

$$= \frac{9}{8}(1 - \frac{c}{2})^2$$

agent's conditional expected utility:

in equilibrium, $v_i + v_j \geq \frac{3}{4}c + \frac{1}{2} \Rightarrow v_j \geq \frac{3}{4}c + \frac{1}{2} - v_i$

over values of v_j:

$$\mathbb{E}U_i = \int_{\frac{3}{4}c+\frac{1}{2}-v_i}^{1} \left[v_i - b_i(v_i) + \frac{1}{2}(b_i(v_i) + b_j(v_j) - c\right] f(v_j)\, dv_j$$

$$= \int_{\frac{3}{4}c+\frac{1}{2}-v_i}^{1} \left[v_i - \left(\frac{c}{4} - \frac{1}{6} + \frac{2}{3}v_i\right) + \frac{1}{2}\left(\frac{c}{4} - \frac{1}{6} + \frac{2}{3}v_i + \frac{c}{4} - \frac{1}{6} + \frac{2}{3}v_j - c\right)\right] f(v_j)\, dv_j$$

$$= \int_{\frac{3}{4}c+\frac{1}{2}-v_i}^{1} \left(\frac{2}{3}v_i - \frac{1}{2}c + \frac{1}{3}v_j\right) dv_j$$

$$= \frac{1}{32}(4v_i - 3c + 2)^2$$

over values of v_i in equilibrium:

$$\mathbb{E}U_i = \int_{\frac{3}{4}c-\frac{1}{2}}^{1} \frac{1}{32}(4v_i - 3c + 2)^2 \; f(v_i)\, dv_i$$

$$= \frac{9}{16}\left(1 - \frac{c}{2}\right)^3$$

Comparison of equilibriums:

	$\mathbb{E}U$	
threshold strategy:	$\frac{1}{2}\left(1-\frac{c}{2}\right)^3$	
linear strategy:	$\frac{9}{16}\left(1-\frac{c}{2}\right)^3$	← better off.

Bilateral trade and double auction

Set-up: Single buyer and seller, both have private information about their valuations of a common good, $v_s, v_b \sim \text{Unif}(0,1)$.

- each submits a sealed bid (seller → asking price, buyer → offer price)
- if $p_b \geq p_s$, trade at some agreed-upon price
- assume linear strategies, i.e. $p_i = \alpha_i + \beta_i v_i$, $i = s, b$.

1. trading price $p = \frac{1}{2}(p_b + p_s)$:

start with buyer's cond. exp. utility:

$$\mathbb{E}U_b(x, p_s \mid v_b) = \int_?^? \left(v_b - \tfrac{1}{2}(x + p_s)\right) f(v_s)\, dv_s.$$

constraint : $x \geq p_s \Rightarrow x \geq \alpha_s + \beta_s v_s$

$\Rightarrow v_s \leq \frac{1}{\beta_s}(x - \alpha_s)$

bounds for integration are $v_s \in [0, \frac{1}{\beta_s}(x-\alpha_s)]$:

$$\mathbb{E}U_b = \int_0^{\frac{1}{\beta_s}(x-\alpha_s)} \left(v_b - \tfrac{1}{2}x - \tfrac{1}{2}\alpha_s - \tfrac{1}{2}\beta_s v_s\right) dv_s$$

$$= \left(v_b - \tfrac{1}{2}x - \tfrac{1}{2}\alpha_s\right)\left(\tfrac{1}{\beta_s}\right)(x - \alpha_s) - \tfrac{1}{2}\beta_s\left(\tfrac{1}{2}\right)\left(\tfrac{1}{\beta_s}\right)^2 (x - \alpha_s)^2$$

$$= \tfrac{1}{\beta_s}\left(v_b - \tfrac{1}{2}x - \tfrac{1}{2}\alpha_s\right)(x - \alpha_s) - \tfrac{1}{4}\left(\tfrac{1}{\beta_s}\right)(x - \alpha_s)^2$$

FOC wrt x: $\frac{1}{\beta_s}\left(v_b - \frac{1}{2}x - \frac{1}{2}\alpha_s\right) + \left(-\frac{1}{2}\right)\left(\frac{1}{\beta_s}\right)(x - \alpha_s) - \frac{1}{2}\left(\frac{1}{\beta_s}\right)(x - \alpha_s) = 0$

$\Rightarrow v_b - \frac{1}{2}x - \frac{1}{2}\alpha_s - x + \alpha_s = 0$

$\Rightarrow x = \frac{2}{3}v_b + \frac{1}{3}\alpha_s$

$x = p_b = \alpha_b + \beta_b v_b \Rightarrow \beta_b = \frac{2}{3}$, $\alpha_b = \frac{1}{3}\alpha_s$

Similarly for seller: trade when $p_b \geq x \Rightarrow \alpha_b + \beta_b v_b \geq x \Rightarrow v_b \geq \frac{1}{\beta_b}(x - \alpha_b)$

$$\therefore \mathbb{E}U_s(p_b, p_s \mid v_s) = \int_{\frac{1}{\beta_b}(x-\alpha_b)}^{1} \left[\tfrac{1}{2}(p_b + x) - v_s\right] f(v_b)\, dv_b$$

$$= \int_{\frac{1}{\beta_b}(x-\alpha_b)}^{1} \left[\tfrac{1}{2}\alpha_b + \tfrac{1}{2}\beta_b v_b + \tfrac{1}{2}x - v_s\right] dv_b$$

$$= \left(\tfrac{1}{2}x + \tfrac{1}{2}\alpha_b - v_s\right)\left(1 - \tfrac{1}{\beta_b}(x - \alpha_b)\right) + \tfrac{1}{2}\beta_b\left(\tfrac{1}{2}\right)\left(1 - \tfrac{1}{\beta_b}(x - \alpha_b)\right)^2$$

FOC wrt x: $\frac{1}{2}\left(1 - \frac{1}{\beta_b}(x - \alpha_b)\right) + \left(\frac{1}{2}x + \frac{1}{2}\alpha_b - v_s\right)\left(-\frac{1}{\beta_b}\right) - \frac{\beta_b}{4}\left(\frac{2}{\beta_b}\right)(x - \alpha_b)\left(-\frac{1}{\beta_b}\right) = 0$

$\Rightarrow -\frac{1}{2}x - \frac{1}{2}\alpha_b + v_s + \frac{1}{2}\beta_b - \frac{1}{2}x + \frac{1}{2}\alpha_b - \frac{1}{2}x + \frac{1}{2}\alpha_b = 0$

$\Rightarrow v_s - \frac{3}{2}x + \frac{1}{2}\alpha_b + \frac{1}{2}\beta_b = 0$

$\Rightarrow x = \frac{2}{3}v_s + \frac{1}{3}(\alpha_b + \beta_b)$

$x = p_s = \alpha_s + \beta_s v_s \Rightarrow \beta_s = \frac{2}{3}$, $\alpha_s = \frac{1}{3}\alpha_b + \frac{1}{3}\beta_b$

$\Rightarrow \alpha_s = \frac{1}{3}\left(\frac{1}{3}\alpha_s\right) + \frac{1}{3}\left(\frac{2}{3}\right) = \frac{1}{4}$

$\Rightarrow \alpha_b = \frac{1}{12}$

In summary, the equilibrium strategy is:

$p_s(v_s) = \frac{1}{4} + \frac{2}{3}v_s$ $\quad$ $p_b(v_b) = \frac{1}{12} + \frac{2}{3}v_b$

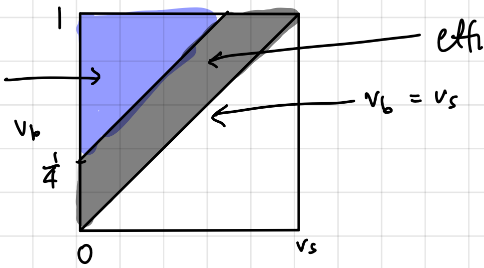
Prob(trade) : $p_b \geq p_s \Rightarrow \frac{1}{12} + \frac{2}{3} v_b \geq \frac{1}{4} + \frac{2}{3} v_s$

$\Rightarrow v_b - v_s \geq \frac{1}{4}$

Prob(trade)

$= \frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)$

$= \frac{9}{32}$



efficiency loss resulting from information asymmetry

2. trading price $p = k p_b + (1-k) p_s$

start with buyer:

- trade occurs if $p_b \geq p_s = \alpha_s + \beta_s v_s \Rightarrow v_s \leq \frac{p_b - \alpha_s}{\beta_s}$

→ buyer gets $v_b - (k p_b + (1-k) p_s(v_s))$

$= v_b - (k p_b + (1-k)(\alpha_s + \beta_s v_s))$

← unknown r.v.

trade occurs for $v_s \in \left[0, \frac{p_b - \alpha_s}{\beta_s}\right]$

so buyer's conditional expected utility:

$$U_b(p_b, p_s \mid v_b) = \int_0^{\frac{p_b - \alpha_s}{\beta_s}} \left[v_b - (k p_b + (1-k)(\alpha_s + \beta_s v_s))\right] f(v_s)\, dv_s$$

$$= \int_0^{\frac{p_b - \alpha_s}{\beta_s}} \left[v_b - k p_b - (1-k)\alpha_s - (1-k)\beta_s v_s\right] f(v_s)\, dv_s$$

$$= \frac{p_b - \alpha_s}{\beta_s}(v_b - k p_b - (1-k)\alpha_s) - \frac{(1-k)\beta_s}{2}\left(\frac{p_b - \alpha_s}{\beta_s}\right)^2$$

FOC wrt p_b : $\frac{1}{\beta_s}(v_b - k p_b - (1-k)\alpha_s) + \frac{p_b - \alpha_s}{\beta_s}(-k) - ((1-k)\beta_s\left(\frac{p_b - \alpha_s}{\beta_s}\right)\left(\frac{1}{\beta_s}\right) = 0$

$$\Rightarrow \frac{1}{\beta_s}(v_b - k p_b - (1-k)\alpha_s - k p_b + k\alpha_s - (1-k) p_b + (1-k)\alpha_s) = 0$$

$$\Rightarrow \frac{1}{\beta_s}(v_b - 2k p_b + k\alpha_s - p_b + k b_p) = 0$$

$$\Rightarrow -\frac{1}{\beta_s}(p_b + k p_b - k\alpha_s - v_b) = 0$$

$$\Rightarrow p_b = \frac{v_b + k\alpha_s}{k+1} = \frac{1}{k+1} v_b + \frac{k}{k+1}\alpha_s$$

Now, for seller: trade at $p_b = \alpha_b + \beta_b v_b \geq p_s \Rightarrow v_b \geq \frac{p_s - \alpha_b}{\beta_b}$

$$U_s(p_b, p_s \mid v_s) = \int_{\frac{p_s - \alpha_b}{\beta_b}}^{1} \left[(k p_b(v_b) + (1-k) p_s) - v_s\right] f(v_b)\, dv_b$$

$$= \int_{\frac{p_s - \alpha_b}{\beta_b}}^{1} \left[(k(\alpha_b + \beta_b v_b) + (1-k) p_s) - v_s\right] dv_b$$

$$= \int_{\frac{p_s - \alpha_b}{\beta_b}}^{1} \left[k\alpha_b + (1-k) p_s - v_s + k\beta_b v_b\right] dv_b$$

$$= \left(1 - \frac{p_s - \alpha_b}{\beta_b}\right)(k\alpha_b + (1-k) p_s - v_s) + k\beta_b\left(\frac{1}{2}\right)\left(1 - \frac{p_s - \alpha_b}{\beta_b}\right)^2$$

MECHANISM DESIGN

$$U_s(p_b, p_s \mid v_s) = \left(1 - \frac{p_s - \alpha_b}{\beta_b}\right)\left(k\alpha_b + (1-k)p_s - v_s\right) + k\beta_b\left(\tfrac{1}{2}\right)\left(1 - \frac{p_s - \alpha_b}{\beta_b}\right)^2$$

FOC wrt p_s: $-\frac{1}{\beta_b}\left(k\alpha_b + (1-k)p_s - v_s\right) + \left(1 - \frac{p_s - \alpha_b}{\beta_b}\right)(1-k) + k\beta_b\left(1 - \frac{p_s - \alpha_b}{\beta_b}\right)\left(-\frac{1}{\beta_b}\right) = 0$

$$\Rightarrow \frac{1}{\beta_b}\left(\alpha_b - k\alpha_b - 2p_s + kp_s + v_s + \beta_b - k\beta_b\right) = 0$$

$$\Rightarrow p_s = \frac{1}{2-k}v_s + \frac{(1-k)(\alpha_b + \beta_b)}{2-k}$$

Solving simultaneously: $\beta_b = \frac{1}{k+1}$, $\quad \alpha_b = \frac{k}{k+1}\alpha_s$

$\beta_s = \frac{1}{2-k}$, $\quad \alpha_s = \frac{1-k}{2-k}(\alpha_b + \beta_b)$

$$= \frac{1-k}{2-k}\left(\frac{k}{k+1}\alpha_s + \frac{1}{k+1}\right)$$

$$= \frac{1-k}{(2-k)(k+1)}(1 + k\alpha_s)$$

$$\Rightarrow \alpha_s = \tfrac{1}{2}(1-k), \quad \alpha_b = \frac{k(1-k)}{2(k+1)}$$

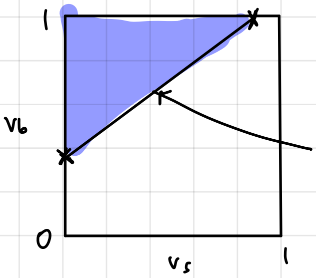
Therefore strategies in equilibrium:

$p_s = \frac{1}{2}(1-k) + \frac{1}{2-k}v_s \qquad p_b = \frac{k(1-k)}{2(k+1)} + \frac{1}{k+1}v_b$

Prob(trade): $p_b \geq p_s \Rightarrow \frac{k(1-k)}{2k+2} + \frac{1}{k+1}v_b \geq \frac{1-k}{2} + \frac{1}{2-k}v_s$

$$\Rightarrow v_b \geq \frac{1}{4-2k}\left(2v_s(1+k) - 3k + k^2 + 2\right)$$

$$\Rightarrow v_b \geq \frac{1}{4-2k}\left(2v_s(1+k) + (k-1)(k-2)\right)$$



$v_b = \frac{1}{4-2k}\left(2v_s(1+k) + (k-1)(k-2)\right)$

Find the boundary points so we can calculate the area of the triangle:

$$v_b = \begin{cases} \frac{1}{2}(1-k) & \text{if } v_s = 0 \\ 1 & \text{if } v_s = 1 - \frac{1}{2}k \end{cases}$$

For what k is Prob(trade) maximized?

$$\text{Prob(trade)} = \tfrac{1}{2}\left(1 - \tfrac{1}{2}(1-k)\right)\left(1 - \tfrac{1}{2}k\right) = -\tfrac{1}{8}k^2 + \tfrac{1}{8}k + \tfrac{1}{4}$$

FOC wrt k: $-\frac{1}{4}k + \frac{1}{8} = 0$

$$\Rightarrow k^* = \tfrac{1}{2}$$

BNE for general case of auctions

Set-up:

- $N = \{1, 2, \dots, n\}$ risk-neutral buyers
- buyers' valuations are drawn from pdf $f(v_i)$, cdf $F(v_i)$, $v_i \in [0,1]$
- buyers have quasi-linear utility $u_i = v_i - b$ if i wins and pays bid b

1. Seller with zero reservation price:
   - i.e. seller wishes to sell object and make the maximum amount of revenue

1st price sealed bid auction:

Agents have a bidding function $b_i : v_i \in [0,1] \to \mathbb{R}^+ = [0, \infty)$. Note that this is strictly increasing, higher valuation ⇒ higher bid.

We seek a symmetric BNE: cond. $\mathbb{E}U_i$ is maximized when i bids $\hat{b}(v_i)$ given others employ the same bidding function $\hat{b}(\cdot)$.

Apply the Revelation Principle: Have each agent report v_i, and the winner is the one who submits the highest report. If i wins by submitting w_i, they pay $\hat{b}(w_i)$.

→ the bidding strategy $\hat{b}(\cdot)$ is BNE ⟺ truth-telling in the described direct mechanism (i.e. $w_i = v_i$) is BNE.

Let $u(w, v)$ be a player's conditional exp. utility when they are of type v, and reports a value of w, and all other players report truthfully:

$$u(w,v) = (v - \hat{b}(w)) \cdot \text{Prob}(i \text{ wins})$$
$$= (v - \hat{b}(w)) \cdot \text{Prob}(w > \text{all other } v)$$
$$= (v - \hat{b}(w)) \cdot \prod_{j \neq i} F_j(w)$$
$$= (v - \hat{b}(w)) \cdot F^{n-1}(w)$$

truth-telling BNE ⟺ $\left.\dfrac{\partial u(w,v)}{\partial w}\right|_{w=v} = 0$

FOC wrt w: $(v - \hat{b}(w)) \cdot (n-1) F^{n-2}(w) \; f(w) + (-\hat{b}'(w)) \cdot F^{n-1}(w) = 0$

Sub $w = v$:

$$(v - \hat{b}(v)) \cdot (n-1) F^{n-2}(v) \, f(v) - \hat{b}'(v) \cdot F^{n-1}(v) = 0$$

$$\Rightarrow v\,(n-1) F^{n-2}(v)\, f(v) = \underbrace{\hat{b}(v)\,(n-1) F^{n-2}(v)\, f(v) + \hat{b}'(v) \cdot F^{n-1}(v)}_{\text{notice that this is expansion of product rule of } \hat{b}(v) F^{n-1}(v) \text{ wrt } v}$$

$$\Rightarrow v\,(n-1)\, F^{n-2}(v)\, f(v) = \frac{\partial\, \hat{b}(v)\, F^{n-1}(v)}{\partial v}$$

Integrate both sides: (let $v = x$)

$$\Rightarrow \int_0^v x\,(n-1)\, F^{n-2}(x)\, f(x)\, dx = \int_0^v \frac{\partial\, \hat{b}(v)\, F^{n-1}(v)}{\partial v}\, dx$$

$$\int_0^v x(n-1)F^{n-2}(x)f(x)\,dx = \int_0^v \frac{\partial \hat{b}(v)F^{n-1}(v)}{\partial v}\,dx$$

$$\Rightarrow \int_0^v x(n-1)F^{n-2}(x)f(x)\,dx = \hat{b}(v)F^{n-1}(v) - \underbrace{\hat{b}(0)}_{=0}F^{n-1}(0) \quad \text{since} \quad 0 \le \hat{b}(v) \le v$$

Thus, we have the equilibrium bidding function: (for $v > 0$)

$$\hat{b}(v) = \frac{1}{F^{n-1}(v)}\int_0^v x(n-1)F^{n-2}(x)f(x)\,dx$$

$$\equiv \frac{n-1}{F^{n-1}(v)}\int_0^v xF^{n-2}(x)f(x)\,dx$$

$$\equiv \frac{1}{F^{n-1}(x)}\int_0^v x\,dF^{n-1}(x)$$

let's check this formula with the uniform distribution, i.e. $f(x) = 1$, $F(x) = x$:

$$\hat{b}(v) = \frac{n-1}{F^{n-1}(v)}\int_0^v xF^{n-2}(x)f(x)\,dx$$

$$= \frac{n-1}{v^{n-1}}\int_0^v x \cdot x^{n-2}\,dx$$

$$= \frac{n-1}{v^{n-1}} \cdot \frac{1}{n}v^n$$

$$= \frac{n-1}{n}v \quad ✓ \qquad \text{for } n = 2,\ \hat{b}(v) = \tfrac{1}{2}v \text{ as shown before}$$

expected revenue: seller collects $\hat{b}(v)$ from bidder i of type v when $v > v_j \ \forall j \neq i$.

$$\mathbb{E}R = \hat{b}(v) \cdot \text{Prob}(i \text{ wins})$$

$$= \hat{b}(v) \cdot F^{n-1}(v) \quad \text{given type } v$$

$$= \int_0^1 \hat{b}(v) \cdot F^{n-1}(v)f(v)\,dv, \quad \text{given bidder } i \text{ with varying } v.$$

in total: $n\int_0^1 \hat{b}(v) \cdot F^{n-1}(v)f(v)\,dv$

$$= n\int_0^1 \frac{n-1}{F^{n-1}(v)}\left[\int_0^v xF^{n-2}(x)f(x)\,dx\right] \cdot F^{n-1}(v)f(v)\,dv$$

$$= n(n-1)\int_0^1\int_0^v xF^{n-2}(x)f(x)\,dx\,f(v)\,dv$$

Revenue equivalence w 2nd price auction:

In 2nd price auction, $\hat{b}(v) = v$.

expected revenue: seller collects v from winner, if v is 2nd highest value

$$\mathbb{E}R = v \cdot (1 - F(v))(F(v))^{n-2} \cdot (n-1) \quad \text{for a given } v$$

$$= \underset{\substack{\nearrow \\ \text{total}}}{n}\underset{\substack{\uparrow \\ \text{possible winners}}}{(n-1)}\int_0^1 v(1 - F(v))F^{n-2}(v)f(v)\,dv$$

We want to show that $n(n-1)\int_0^1\int_0^v xF^{n-2}(x)f(x)\,dx\,f(v)\,dv$ ← 1st price $\mathbb{E}R$

$$=$$

$n(n-1)\int_0^1 v(1 - F(v))F^{n-2}(v)f(v)\,dv$ ← 2nd price $\mathbb{E}R$

from 1st price ER: $n(n-1) \int_0^1 \int_0^v x F^{n-2}(x) f(x)\, dx\, f(v)\, dv$

$$= n(n-1) \int_0^1 x F^{n-2}(x) f(x) \int_x^1 f(v)\, dv\, dx$$

$$= n(n-1) \int_0^1 x F^{n-2}(x) f(x) [F(1) - F(x)]\, dx$$

$$= n(n-1) \int_0^1 x F^{n-2}(x) f(x) [1 - F(x)]\, dx$$

replace $x = v$:

$$= n(n-1) \int_0^1 v (1 - F(v)) F^{n-2}(v) f(v)\, dv$$

$\equiv$ 2nd price ER. $\blacksquare$

2. seller with reservation price (i.e. there is a minimum bid that the seller will only trade at)
- now, x is bounded from below by reservation price b_0

At reservation price b_0:
- $\hat{b}(v) \geq b_0$
- $\hat{b}(b_0) = b_0$, if bidder's value matches res price, will bid that (bidding higher leads to -ve utility, bidding lower $\Rightarrow$ no trade)

Applying the formula from before: (note lower bound of integral is b_0)

$$\int_{b_0}^v x (n-1) F^{n-2}(x) f(x)\, dx = \hat{b}(v) F^{n-1}(v) - \hat{b}(b_0) F^{n-1}(b_0)$$

$$= \hat{b}(v) \cdot F^{n-1}(v) - b_0 \cdot F^{n-1}(b_0)$$

e.g. $n=2$, uniform dist, $f(v) = 1$, $F(v) = v$

$$\int_{b_0}^v x (n-1) F^{n-2}(x) f(x)\, dx = \int_{b_0}^v x\, dx$$

$$= \tfrac{1}{2} v^2 - \tfrac{1}{2} b_0^2$$

$$\hat{b}(v) \cdot F^{n-1}(v) - b_0 \cdot F^{n-1}(b_0) = \hat{b}(v) \cdot v - b_0 \cdot b_0$$

$$= v \hat{b}(v) - b_0^2$$

$$\Rightarrow \tfrac{1}{2} v^2 - \tfrac{1}{2} b_0^2 = v \hat{b}(v) - b_0^2$$

$$\Rightarrow \hat{b}(v) = \frac{v^2 + b_0^2}{2v}$$

exp. revenue:

1st price auction: $R_{FSA} = 2 \int_{b_0}^1 \frac{v^2 + b_0^2}{2v} F(v) f(v)\, dv$

$$= \int_{b_0}^1 (v^2 + b_0^2)\, dv$$

$$= \tfrac{1}{3} v^3 - \tfrac{4}{3} b_0^3 + b_0^2$$

FOC wrt b_0: $-4 b_0^2 + 2 b_0 = 0$

$\Rightarrow b_0 = \frac{1}{2}$ to maximize revenue.

$$R_{FSA}^* = \tfrac{1}{3} - \tfrac{4}{3} (\tfrac{1}{2})^3 + \tfrac{1}{2}^2 = \frac{5}{1}$$

MECHANISM DESIGN

Apr 9

Optimal auction design for a single object

- Revelation principle in this setting:
  1. Consider any mechanism
  2. look at an equilibrium outcome of this mechanism : allocation of the object ; payments.
  3. Determine the payoffs : expected utility of each buyer and the expected revenue of the seller
  4. Then there is a direct mechanism with a truth-telling equilibrium that achieves the same outcome and payoffs.

→ Search for the best direct mechanism with a truth-telling equilibrium i.e. incentive compatible direct mechanism.

^ what does this look like?

→ Finally, optimization.

Set-up :

- n bidders : $N = \{1, 2, \dots, n\}$
- seller has a single unit to sell
- $v_i \sim [\underline{v}, \bar{v}]$ with $F_i(v_i)$ and $f_i(v_i)$
- quasi-linear utility / risk-neutral buyers
- direct mechanism : ask agents to report their valuations. Let w_i be i's reported value.
  - $p_i(w_1, w_2, \dots, w_n)$ is the probability that i wins the object note that $\sum_{i=1}^{n} p_i \leq 1$.
  - $r_i(w_1, w_2, \dots, w_n)$ is the payment i makes to the seller = revenue that seller collects.

outcomes

equilibrium :

- $U_i(w_i, v_i)$: buyer i's conditional expected utility by reporting w_i when true value is v_i, assuming others report truthfully.
- truth-telling (incentive compatibility) is a Bayesian Nash eq :

  $U_i(v_i, v_i) \geq U_i(w_i, v_i) \quad \forall\ i = 1, 2, \dots, n$ and $v_i \in [\underline{v}, \bar{v}]$.

some notation:

- (w_i, v_{-i}) : everyone except i tells the truth

  i.e. $(v_1, v_2, \dots, v_{i-1}, w_i, v_{i+1}, \dots, v_n)$

- $\prod_{i \neq j} f_j(v_j)\, dv_j = f_1(v_1) \cdot \dots \cdot f_{i-1}(v_{i-1})\, f_{i+1}(v_{i+1}) \cdots f_n(v_n) \cdot dv_1 \cdots dv_{i-1}\, dv_{i+1} \cdots dv_n$

  $= f_{-i}(v_{-i})\, dv_{-i}$

  ↖ we need this for when we calculate expectations over unknown values of all agents other than i

- $U_i(w_i, v_i) = \int \cdots \int \underbrace{[v_i \underbrace{p_i(w_i, v_{-i})}_{\text{prob of i win, given report } w_i \text{ and everyone else telling the truth}} - \overset{\text{payment to seller}}{r_i(w_i, v_{-i})}]}_{\text{net utility, function of } v_{-i} \text{ which are unknowns}} f_{-i}(v_{-i})\, dv_{-i}$

  (cond. exp. utility)

pull it out, since it's a constant!

$$= v_i \int \cdots \int p_i(w_i, v_{-i})\, f_{-i}(v_{-i})\, dv_{-i} - \int \cdots \int r_i(w_i, v_{-i})\, f_{-i}(v_{-i})\, dv_{-i}$$

$$= v_i \underbrace{\bar{p}_i(w_i)}_{\text{prob of winning when reporting } w_i} - \underbrace{R(w_i)}_{\text{expected payment i makes when reporting } w_i}$$

when i tells the truth,

$U_i(v_i, v_i) = U_i(v_i) = v_i \bar{p}_i(v_i) - R(v_i)$

if i is type w_i : $U_i(w_i) = w_i \bar{p}(w_i) - R(w_i)$

Hence if type v_i mimics w_i, ~~them~~ their utility will be:

$$U_i(w_i, v_i) = v_i \bar{p}_i(w_i) - R_i(w_i)$$
$$= v_i \bar{p}_i(w_i) + U_i(w_i) - w_i \bar{p}(w_i)$$
$$= U_i(w_i) + (v_i - w_i)\, \bar{p}_i(w_i)$$

(cond. exp. for type w_i plus some incremental)

truth-telling eq implies that v_i has no incentive to mimic w_i,

i.e. $U_i(w_i, v_i) \leq U_i(v_i)$

$\Rightarrow \quad U_i(w_i) + (v_i - w_i)\, \bar{p}_i(w_i) \leq U_i(v_i) \quad \forall\, w_i$

MECHANISM DESIGN

Let $w_i = v_i + dv_i$ (truth and some deviation from it)

Then: $u_i(v_i) \geq u_i(v_i + dv_i) - \bar{p}_i(v_i + dv_i)\, dv_i$

Similarly, for type $v_i + dv_i$ not to mimic v_i, we have:

$$u_i(v_i + dv_i) \geq u_i(v_i) + \bar{p}_i(v_i)\, dv_i$$

No matter what the truth is, agents will not lie at the margins!
Hence we put the restrictions in both directions.

Both eqs:

$$u_i(v_i) + \bar{p}_i(v_i + dv_i)\, dv_i \geq u_i(v_i + dv_i) \geq u_i(v_i) + \bar{p}_i(v_i)\, dv_i$$

$$\Rightarrow \bar{p}_i(v_i + dv_i)\, dv_i \geq u_i(v_i + dv_i) - u_i(v_i) \geq \bar{p}_i(v_i)\, dv_i \quad (*)$$

In (*) we have bounded the utility difference. Note that both bounds have the term dv_i

If $dv_i > 0$, then $\bar{p}_i(v_i + dv_i) \geq \bar{p}_i(v_i)$ follows. This suggests that $\bar{p}_i(\cdot)$ is non-decreasing.

Moreover, $\bar{p}_i(v_i + dv_i) \geq \dfrac{u_i(v_i + dv_i) - u_i(v_i)}{dv_i} \geq \bar{p}_i(v_i)$

Let $dv_i \to 0$. Then we have $\dfrac{d u_i(v_i)}{dv_i} = \bar{p}_i(v_i)$

Integrate: $u_i(v_i) = u_i(\underline{v}) + \int_{\underline{v}}^{v_i} \bar{p}_i(x)\, dx$

Recall that $u_i(v_i) = v_i \bar{p}_i(v_i) - R_i(v_i)$

Hence we also have

$$\begin{aligned} R_i(v_i) &= v_i \bar{p}_i(v_i) - u_i(v_i) \\ &= v_i \bar{p}_i(v_i) - u_i(\underline{v}) - \int_{\underline{v}}^{v_i} \bar{p}_i(x)\, dx \end{aligned}$$

↑ cond. exp. utility when i's type is the lowest possible value.

↑ is prob of winning when reporting diff type

Revenue Equivalence Theorem: Take any 2 mechanisms. As long as they have the same $u_i(\underline{v})$ and $\bar{p}_i(v_i)$ $\forall i$, then the seller obtains the same revenue from type v_i of i and hence the same expected revenue.

$\bar{p}_i(\underline{v}) = 0 \Rightarrow R_i(\underline{v}) = -U_i(\underline{v})$

(prob of lowest type winning)

$$\Rightarrow R_i(v_i) = v_i \underset{\uparrow}{\bar{p}_i(v_i)} + \underset{\uparrow}{R_i(\underline{v})} - \int_{\underline{v}}^{v_i} \bar{p}_i(x)\,dx$$

these are expectations.

Recall: $U_i(v_i) = U_i(\underline{v}) + \underbrace{\int_{\underline{v}}^{v_i} \bar{p}_i(x)\,dx}_{\text{non-negative.}}$

Individual rationality / participation constraint:

$U_i(v_i) \geq 0$ for any v_i

$\Leftrightarrow U_i(\underline{v}) \geq 0$

$\Leftrightarrow R_i(\underline{v}) \leq 0$

In summary so far we have that:

- $\bar{p}_i(v_i)$ is a weakly increasing function
- $U_i(w_i, v_i) = v_i \bar{p}_i(w_i) - R_i(w_i)$
- $R_i(v_i) = v_i \bar{p}_i(v_i) + R_i(\underline{v}) - \int_{\underline{v}}^{v_i} \bar{p}_i(x)\,dx$
-

If type v_i mimics w_i, then their utility will be:

$$U_i(w_i, v_i) = v_i \bar{p}(w_i) - R_i(w_i)$$

$$= v_i \bar{p}_i(w_i) - w_i \bar{p}_i(w_i) - \underset{\uparrow \text{ a constant}}{R_i(\underline{v})} + \int_{\underline{v}}^{w_i} \bar{p}_i(x)\,dx$$

FOC wrt w_i:

$$v_i \bar{p}_i'(w_i) - [w_i \bar{p}_i'(w_i) + \bar{p}_i(w_i)] + \bar{p}_i(w_i) = 0$$

$$\Rightarrow v_i \bar{p}_i'(w_i) - w_i \bar{p}_i'(w_i) = 0$$

$$\Rightarrow (v_i - w_i) \bar{p}_i'(w_i) = 0$$

↖ $\bar{p}_i$ weakly increasing!

$$\Rightarrow v_i = w_i$$

truth-telling is optimal ✓

Apr 14

Incentive compatibility in terms of conditional EU:

$$U_i(v_i) = U_i(\underline{v}) + \int_{\underline{v}}^{v_i} \bar{p}_i(x)\,dx$$

↖ cond. exp. utility of lowest type

↖ integral of exp winning prob from lowest possible valuation to v_i we are considering

MECHANISM DESIGN

Incentive compatibility in terms of cond. exp. revenue and $U_i(\underline{v})$:

$$R_i(v_i) = v_i\,\bar{p}_i(v_i) - U_i(v_i)$$
$$= v_i\,\bar{p}_i(v_i) - U_i(\underline{v}) - \int_{\underline{v}}^{v_i} \bar{p}_i(x)\,dx$$

~~[illegible]~~ $p_i(\underline{v}) = 0 \Rightarrow R_i(\underline{v}) = -U_i(\underline{v})$

$$\therefore\ R_i(v_i) = v_i\,\bar{p}_i(v_i) + R_i(\underline{v}) - \int_{\underline{v}}^{v_i} \bar{p}_i(x)\,dx.$$

Individual rationality: recall our defn $U_i(v_i) = U_i(\underline{v}) + \int_{\underline{v}}^{v_i} \bar{p}_i(x)\,dx \geq 0$

$U_i(v_i) \geq 0 \quad \forall\, v_i$

$\Leftrightarrow U_i(\underline{v}) \geq 0$

$\Leftrightarrow R_i(\underline{v}) \leq 0$

Expected Revenue maximization:

- IC #1: $\bar{p}_i(v_i)$ weakly increasing in v_i
- IC #2: $R_i(v_i) = v_i\,\bar{p}_i(v_i) + R_i(\underline{v}) - \int_{\underline{v}}^{v_i} \bar{p}_i(x)\,dx$
- IR: $U_i(v_i) \geq 0\ \forall\, v_i \Leftrightarrow U_i(\underline{v}) \geq 0 \Leftrightarrow R_i(\underline{v}) \leq 0$

$$\Rightarrow \quad \max\ R = \sum_{i\in N} \int_{\underline{v}}^{\bar{v}} R_i(v_i)\, f(v_i)\, dv_i$$

s.t. IC#1, IC#2, IR.

~~using IC#2: $\sum_{i\in N} \int_{\underline{v}}^{\bar{v}} R_i(v_i)\, f_i(v_i)\, dv_i$~~

~~$\Rightarrow \sum_{i\in N} \int_{\underline{v}}^{\bar{v}} (v_i\,\bar{p}_i(v_i) + R_i(\underline{v}) - \int_{\underline{v}}^{v_i} \bar{p}_i(x)\,dx)\, f_i(v_i)\, dv_i$~~

~~let $[\underline{v}, \bar{v}] = [0,1]$ $= \sum_{i\in N} \int_0^1 (v_i\,\bar{p}_i(v_i) + R_i(\underline{v}) - \int_{\underline{v}}^{v_i} \bar{p}_i(x)\,dx)\, f_i(v_i)\, dv_i$~~

~~$= \sum_{i\in N} \int_0^1 (v_i\,\bar{p}_i(v_i)$~~ [illegible]

MECHANISM DESIGN

$$R = \sum_{i \in N} \int_{\underline{v}}^{\bar{v}} R_i(v_i)\, f_i(v_i)\, dv_i$$

let $[\underline{v}, \bar{v}] = [0, 1]$

$$= \sum_{i \in N} \int_0^1 R_i(v_i)\, f_i(v_i)\, dv_i$$

use IC to sub $R_i(v_i)$

$$= \sum_{i \in N} \int_0^1 \left(v \bar{p}_i(v) + R_i(\underline{v}) - \int_{\underline{v}}^{v} \bar{p}_i(x)\, dx \right) f_i(v)\, dv$$

$$= \sum_{i \in N} \int_0^1 \left(v \bar{p}_i(v) + R_i(0) - \int_0^{v} \bar{p}_i(x)\, dx \right) f_i(v)\, dv$$

take out constant

$$= \sum_{i \in N} \int_0^1 \left(v \bar{p}_i(v) - \int_0^{v} \bar{p}_i(x)\, dx \right) f_i(v)\, dv + \sum_{i \in N} R_i(0)$$

expand

$$= \sum_{i \in N} \left[\int_0^1 v \bar{p}_i(v)\, f_i(v)\, dv - \int_0^1 \int_0^{v} \bar{p}_i(x)\, dx\, f_i(v)\, dv \right] + \sum_{i \in N} R_i(0)$$

change order of integrals

$$= \sum_{i \in N} \left[\int_0^1 v \bar{p}_i(v)\, f_i(v)\, dv - \int_0^1 \int_x^1 \bar{p}_i(x)\, f_i(v)\, dv\, dx \right] + \sum_{i \in N} R_i(0)$$

solve

$$= \sum_{i \in N} \left[\int_0^1 v \bar{p}_i(v) f_i(v)\, dv - \int_0^1 \bar{p}_i(x) [1 - F_i(x)]\, dx \right] + \sum_{i \in N} R_i(0)$$

make common element

$$= \sum_{i \in N} \left[\int_0^1 v \bar{p}_i(v) f_i(v)\, dv - \int_0^1 \bar{p}_i(v) [1 - F_i(v)]\, dv \right] + \sum_{i \in N} R_i(0)$$

combine integral

$$= \sum_{i \in N} \left[\int_0^1 \bar{p}_i(v) \Big[\underbrace{v - \frac{1 - F_i(v)}{f_i(v)}}_{\text{"virtual value"}} \Big] f_i(v)\, dv \right] + \sum_{i \in N} R_i(0)$$

"virtual value" = actual value − rent that seller pays due to info asymmetry

$$= \sum_{i \in N} \overbrace{\int_0^1 \cdots \int_0^1}^{n \text{ layers}} p_i(v_1, \dots, v_n) \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] f_1(v_1) \cdot \dots \cdot f_n(v_n)\, dv_1 \dots dv_n + \sum_{i \in N} R_i(0)$$

we use the original defn of $\bar{p}_i$ (expected prob of winning)

$$= \int_0^1 \cdots \int_0^1 \Big\{ \sum_{i \in N} \underline{p_i(v_1, \dots, v_n)} \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] \Big\} f_1(v_1) \dots f_n(v_n)\, dv_1 \dots dv_n + \sum_{i \in N} R_i(0)$$

can view this as a weight; up to seller to allocate. ⟹ want to put highest weight on the highest virtual value.

This is what the seller wishes to maximize.

Solving max R: $$p_i^*(v_1, \dots, v_n) = \begin{cases} 1 & \text{if } v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} > \max\left(v_j - \frac{1 - F_j(v_j)}{f_j(v_j)}, 0\right) \; \forall j \neq i \\ 0 & \text{otherwise.} \end{cases}$$

need to be non-neg.

payments / revenue:

$$R_i^*(v_i) = v_i \bar{p}_i^*(v_i) + R_i^*(0) - \int_0^v \bar{p}_i^*(x)\,dx$$

where $\bar{p}_i^*(v_i) = \int_0^1 \cdots \int_0^1 p_i^*(v_i, v_{-i})\, f_{-i}(v_{-i})\,dv_{-i}$

We seek $r_i^*(v_1, \dots, v_n)$.

Decomposition: $r_i^*(v_i, v_{-i}) = v_i\, p_i^*(v_i, v_{-i}) + r_i^*(0) - \int_0^v p_i^*(x, v_{-i})\,dx$
$(\forall v_i)$

$$R_i^*(v_i) = \int_0^1 \cdots \int_0^1 r_i^*(v_i, v_{-i})\, f_{-i}(v_{-i})\,dv_{-i}$$

some observations:

1. inefficiency: $\sum_{i \in N} p_i^*(v_1, \dots, v_n) < 1$ when for e.g. virtual values
$v_i - \frac{1-F_i(v_i)}{f_i(v_i)} < 0 \quad \forall i$

2. possible that buyer reporting highest value does not win (since we are ranking virtual values, this might be different)
↖ but this will not occur if:
   - symmetric i.e. $F_i = F$, $f_i = f$ $\forall i$
   - virtual value is strictly increasing in v
     e.g. uniform $f(v) = 1$, $F(v) = v$,
     $v - \frac{1-v}{1} = 2v - 1$, strictly increasing v

   i wins (given those 2 conditions) if they have the highest virtual value and $v_i - \frac{1-F(v_i)}{f(v_i)} > 0$ or $v_i > v^*$ where
   $v^* - \frac{1-F(v^*)}{f(v^*)} = 0$
   (uniform dist ⇒ $v^* = \frac{1}{2}$)
